1. Consider a normal 8×8 chessboard, where each square is labelled with either 1 or -1. Let a_k be the product of the numbers in the kth row, and let b_k be the product of the numbers in the kth column. Find, with proof, all possible values of $\sum_{k=1}^{8}(a_k b_k)$.

   SOLUTION

   The possible values of $\sum_{k=1}^{8} a_k b_k$ are $0, \pm 4, \pm 8$. To justify the answer, we need two steps.

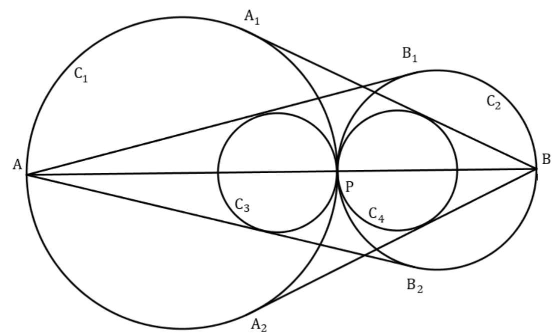
   Step 1. **We show that these values are possible. In fact if the first row of the chessboard has zero, two, four, six, or eight -1's, and numbers in all other squares are 1's, then $\sum_{k=1}^{8} a_k b_k$ takes the values 8, 4, 0, -4, and -8, respectively (there are of course other ways to realize these numbers).**

   Step 2. **We show that these are the only possible values. It is clear that a_k, b_k are either 1 or -1, so is $a_k b_k$, therefore $-8 \le \sum_{k=1}^{8} a_k b_k \le 8$. It remains to show that $\sum_{k=1}^{8} a_k b_k$ is a multiple of 4. We give two proofs:**

   Method 1. We observe that $\prod_{k=1}^{8}(a_k b_k)$ is always 1 because the number (either 1 or -1) in each square of the chessboard appears twice in the product. Therefore there are even number of -1's among $a_1 b_1, \ldots, a_8 b_8$. Let the number of -1's be $2k$ with k a nonnegative integer. Then there are $8-2k$ 1's among $a_1 b_1, \ldots, a_8 b_8$. So $\sum_{k=1}^{8} a_k b_k = (8-2k)(1) + (2k)(-1) = 8 - 4k = 4(2-k)$, a multiple of 4.

   Method 2. For a given labeling/configuration of the chessboard, we study the change of $\sum_{k=1}^{8} a_k b_k$ if we change the number at the ith row and the jth column to its opposite $(1 \le i, j \le 8)$. In this case, both a_i and b_j become their respective opposites and other a_k, b_k are unchanged. If $i = j$, then $a_i b_i$ does not change and $\sum_{k=1}^{8} a_k b_k$ does not change. If $i \ne j$, then $a_i b_i$ and $a_j b_j$ are changed to their opposites. If both $a_i b_i$ and $a_j b_j$ are 1's, then $\sum_{k=1}^{8} a_k b_k$ becomes 4 less; If both $a_i b_i$ and $a_j b_j$ are -1's, then $\sum_{k=1}^{8} a_k b_k$ becomes 4 more; If one of $a_i b_i$ and $a_j b_j$ is 1, and the other is -1, then $\sum_{k=1}^{8} a_k b_k$ is unchanged; In summary, if we alter one number on the chessboard to its opposite, $\sum_{k=1}^{8} a_k b_k$ is altered by a multiple of 4. If all the numbers on the chessboard are 1's (call this the basic configuration), then $\sum_{k=1}^{8} a_k b_k = 8$, a multiple of 4. For any given configuration, it can be produced starting from the basic one by altering the number on the squares in finite steps (at most 64 steps), and at each step $\sum_{k=1}^{8} a_k b_k$ is altered by a multiple of 4, so $\sum_{k=1}^{8} a_k b_k$ is also a multiple of 4 for this given configuration.

2. Let $\overline{AB}$ be a line segment with $AB = 1$, and P be a point on $\overline{AB}$ with $AP = x$, for some $0 < x < 1$. Draw circles C_1 and C_2 with $\overline{AP}$, $\overline{PB}$ as diameters, respectively. Let $\overline{AB_1}$, $\overline{AB_2}$ be tangent to C_2 at B_1 and B_2, and let $\overline{BA_1}, \overline{BA_2}$ be tangent to C_1 at A_1 and A_2. Now C_3 is a circle tangent to C_2, $\overline{AB_1}$, and $\overline{AB_2}$; C_4 is a circle tangent to C_1, $\overline{BA_1}$, and $\overline{BA_2}$.



(a) Express the radius of C_3 as a function of x.

(b) Prove that C_3 and C_4 are congruent.

SOLUTION

(a) (i) Let the center of the circle C_i be O_i with radius r_i $(i = 1, 2, 3, 4)$. Then $r_2 = PB/2 = (1-x)/2$, $AO_3 = AP - O_3P = x - r_3$, $AO_2 = AP + r_2 = x + \frac{1-x}{2} = \frac{1+x}{2}$. By the similarity of right triangles $\triangle AO_3B_1'$ and $\triangle AO_2B_1$, where B_1' is the point of tangency of $\overline{AB_1}$ to C_3, we have $\frac{AO_3}{AO_2} = \frac{O_3B_1'}{O_2B_1}$, or

$$\frac{x - r_3}{(1+x)/2} = \frac{r_3}{(1-x)/2},$$

which gives $r_3 = \frac{x(1-x)}{2}$ with $0 < x < 1$.

(b) (ii) By symmetry, we can obtain r_4 by changing x to $1 - x$ in the formula for r_3, which gives

$$r_4 = \frac{(1-x)(1-(1-x))}{2} = \frac{(1-x)x}{2} = r_3;$$

hence C_3 and C_4 are congruent.

Remark. This problem is a slight modification of a Japanese Sankaku problem that, according to the book "Sacred Mathematics, Japanese Temple Geometry" by Fukagawa Hidetoshi and Tony Rothman in pages 102–103, was written on a tablet hung in 1842 at the Atsuta shrine of Nagoya City, Aichi prefecture; see this book for more details.

3. Suppose that the graphs of $y = (x + a)^2$ and $x = (y + a)^2$ are tangent to one another at a point on the line $y = x$. Find all possible values of a.

SOLUTION

First note that the quadratic equation $ax^2 + bx + c = 0$ has a double root iff $b^2 = 4ac$.

Also note that if a line is tangent to a parabola then the equation describing their intersection will have a double root.

Since the curves are tangent to each other, their slopes are identical at that point. And since they are reflections of each other around $y = x$, their tangent point will also correspond to the same point on both parabolae, meaning that $\Delta y/\Delta x$ on the tangent line of the first matches $\Delta x/\Delta y$ on the tangent line of the second, meaning the two curves have reciprocal slopes at their tangent point.

Identical slopes that are reciprocals of each other really narrows the field: the slope must be either 1 or -1.

Slope is 1:

In this case both parabolae are tangent to the line $y = x$. They intersect it and have the same slope as it.

So,

$$\begin{aligned} x &= (x+a)^2 \\ x &= x^2 + 2ax + a^2 \\ x^2 + (2a-1)x + a^2 &= 0 \end{aligned}$$

Since it has a double root,

$$\begin{aligned} (2a-1)^2 &= 4a^2 \\ 4a^2 - 4a + 1 &= 4a^2 \\ -4a + 1 &= 0 \\ a &= 1/4 \end{aligned}$$

(x, y) **is** $(1/4, 1/4)$ **at this point of tangency.**

Slope is -1:

This one is harder because we don't know what the tangent line is.

The line tangent to both curves is $y = -x + k$ ... whatever k is.

So,

$$\begin{aligned} -x + k &= (x+a)^2 \\ -x + k &= x^2 + 2ax + a^2 \\ x^2 + (2a+1)x + a^2 - k &= 0 \end{aligned}$$

This must have a double root, so

$$\begin{aligned} (2a+1)^2 &= 4a^2 - 4k \\ 4a^2 + 4a + 1 &= 4a^2 - 4k \\ 4a + 1 &= -4k \\ k &= -a - 1/4 \end{aligned}$$

We also know that the two curves meet on the line $y = x$, so $x = (x + a)^2$.

And the line $y = -x + k$ also meets that same point, so

$$\begin{aligned} x &= -x + k \\ x &= k/2 \\ x &= -a/2 - 1/8 \end{aligned}$$

So, $-a/2 - 1/8 = (-a/2 - 1/8 + a)^2 = (a/2 - 1/8)^2$

Multiplying through by 64,

$$\begin{aligned} -32a - 8 = (4a - 1)^2 &= 16a^2 - 8a + 1 \\ 16a^2 + 24a + 9 &= 0 \\ (4a + 3)^2 &= 0 \\ a &= -3/4 \end{aligned}$$

The possible values of a are, thus, $1/4$ and $-3/4$.

Note that if $a = -3/4$, the curves are both tangent to $y = -x + 1/2$ at the point $(1/4, 1/4)$. Both curves are on the same side of the tangent line and the curves cross each other. If you learned naively in high school (as I did) that tangency means that the curves intersect but do not cross, then you would not consider the curves tangent here. But, of course, "tangent" and "cross" are two completely different concepts. They are tangent AND they cross.

4. You may assume without proof or justification that the infinite radical expressions

$$\sqrt{a-\sqrt{a-\sqrt{a-\sqrt{a-\cdots}}}}$$

and

$$\sqrt{a-\sqrt{a+\sqrt{a-\sqrt{a+\cdots}}}}$$

represent unique values for $a > 2$.

(a) Find a real number a such that

$$\sqrt{a-\sqrt{a-\sqrt{a-\sqrt{a-\cdots}}}} = 2017.$$

(b) Show that

$$\sqrt{2018-\sqrt{2018+\sqrt{2018-\sqrt{2018+\cdots}}}} = \sqrt{2017-\sqrt{2017-\sqrt{2017-\sqrt{2017-\cdots}}}}.$$

SOLUTION

(a) If

$$\sqrt{a-\sqrt{a-\sqrt{a-\sqrt{a-\cdots}}}} = x$$

then

$$\begin{aligned}\sqrt{a-x} &= x\\ a-x &= x^2\\ a &= x^2+x\end{aligned}$$

If $x = 2017$**,** $a = x(x+1) = 2017 \times 2018 = 4070306$**.**

(b) First, consider $\sqrt{a-\sqrt{a-\sqrt{a-\sqrt{a-\cdots}}}}$ **for** $a > 2$**.**

As above, this value must be a root of $x^2 + x - a = 0$**. From the Quadratic Formula,** x **must be one of** $\frac{-1\pm\sqrt{1+4a}}{2}$**. One of these values is negative and so cannot be the value we seek. The other is positive (**$a > 2$ **implies** $x > 1$**) and so must be the value of** $\sqrt{a-\sqrt{a-\sqrt{a-\sqrt{a-\cdots}}}}$**.**

Now, consider $\sqrt{a'-\sqrt{a'+\sqrt{a'-\sqrt{a'+\cdots}}}}$ **where** $a' = a + 1$**. This value must satisfy the equation**
$\sqrt{a'-\sqrt{a'+x}} = x$**, or, after some algebraic manipulation,** $x^4 - 2a'x^2 - x + a'^2 - a' = 0$**.**
Using polynomial division, we see that $\frac{x^4-2a'x^2-x+a'^2-a'}{x^2+x-a} = x^2 - x - a'$ **with a remainder of 0.**
So, $x^4 - 2a'x^2 - x + a'^2 - a' = (x^2 + x - a)(x^2 - x - a')$**.**
From the quadratic formula, the roots of $x^2 - x - a' = 0$ **are** $\frac{1\pm\sqrt{1+4a'}}{2}$**. One of these roots must be negative (since** $a' > 3$**), and as for the other:**

$$
\begin{aligned}
\frac{1+\sqrt{1+4a'}}{2} &> \frac{1+\sqrt{4a'}}{2} \\
&= \frac{1+2\sqrt{a'}}{2} \\
&= 1/2+\sqrt{a'} \\
&> \sqrt{a'}
\end{aligned}
$$

Obviously, $\sqrt{a'-\sqrt{a'+\sqrt{a'-\sqrt{a'+\cdots}}}} < \sqrt{a'}$**, so this root cannot possibly be the value of** $\sqrt{a'-\sqrt{a'+\sqrt{a'-\sqrt{a'+\cdots}}}}$**.**

And, as above, x^2+x-a **has two roots, one of which is negative, and one of which is equal to** $\sqrt{a-\sqrt{a-\sqrt{a-\sqrt{a-\cdots}}}}$**.**

So, of the four roots of $x^4-2a'x^2-x+a'^2-a'=0$**, two are negative, and one is too large, so the one remaining root must be the value of** $\sqrt{a'-\sqrt{a'+\sqrt{a'-\sqrt{a'+\cdots}}}}$**, and that root we already know to be equal to** $\sqrt{a-\sqrt{a-\sqrt{a-\sqrt{a-\cdots}}}}$**.**

And, in this specific case, $a=2017$ **and** $a'=2018$**.**

5. (a) Suppose that m, n are positive integers such that $7n^2 - m^2 > 0$. Prove that, in fact, $7n^2 - m^2 \geq 3$.
   (b) Suppose that m, n are positive integers such that $\frac{m}{n} < \sqrt{7}$. Prove that, in fact, $\frac{m}{n} + \frac{1}{mn} < \sqrt{7}$.

SOLUTION

(a) We are going to evaluate the quantity $7n^2 - m^2$ modulo 7. The integer m can have one of the following forms: $7k$, $7k \pm 1$, $7k \pm 2$, and $7k \pm 3$. This implies that m^2 has one of the forms $7k$, $7k + 1$, $7k + 4$, or $7k + 2$. This means that the reminder modulo 7 of the quantity $7n^2 - m^2$ could be 0 (which means that $7n^2 - m^2$ is at least 7, because of the initial strict inequality), 6, 3, or 5. All these cases give the required $7n^2 - m^2 \geq 3$.

(b) We first notice that the inequality that we have to prove is equivalent, after squaring, to

$$\frac{m^2}{n^2} + 2\frac{1}{n^2} + \frac{1}{m^2n^2} < 7.$$

If $m > 1$, we have

$$\begin{aligned} 7 &\geq \frac{m^2}{n^2} + \frac{3}{n^2} && \textbf{(by part 1)} \\ &= \frac{m^2}{n^2} + 2\frac{1}{n^2} + \frac{1}{n^2} \\ &> \frac{m^2}{n^2} + 2\frac{1}{n^2} + \frac{1}{m^2n^2} && \textbf{(because } m > 1\textbf{).} \end{aligned} \qquad (\star)$$

The case $m = 1$ is trivial: $\frac{m}{n} + \frac{1}{mn} = \frac{1}{n} + \frac{1}{n} = \frac{2}{n} \leq 2 < \sqrt{7}$.

Method 2 for part 1. **We are going to evaluate the quantity $7n^2 - m^2$ modulo 8. We first make the observation that a perfect square a^2 is of the form $8k$ or $8k + 4$ (in the case when a is even), or $8k + 1$ (when a is odd). Consequently $7n^2$ has one of the forms $8k$, $8k + 4$, or $8k + 7$. This means that the reminder modulo 8 of the quantity $7n^2 - m^2$ could be 0 (which means that $7n^2 - m^2$ is at least 8, because of the initial strict inequality), 3, 4, 6 or 7. All these cases give the required $7n^2 - m^2 \geq 3$.**

Note. This problem was proposed by Radu Gologan to a 1978 mathematical contest in Romania.