THE SIXTY-SECOND ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

Part I

Solutions

1. When 2018^{2018} is expanded in base ten, what is the sum of the first digit and the last digit?

   (A) 5 $\boxed{\text{(B) } 6}$ (C) 7 (D) 8 (E) 10

   Solution. The last digit of $2018^{2018} = (2 \cdot 10^3 + 10 + 8)^{2018} = 2^{2018} \cdot (10^3)^{2018} + \cdots + 8^{2018}$ is determined by the last digit of 8^{2018}. The last digit of powers of eight go in a cycle of length four: $8^1 \sim 8$, $8^2 \sim 4$, $8^3 \sim 2$, $8^4 \sim 6$, $8^5 \sim 8$, and so on. Therefore 8^{2018} has a last digit of 4 since 2018 is of the form $4 \cdot (504) + 2$. To decide the first digit of 2018^{2018}, we will consider $\log_{10}(2018^{2018}) = 2018 \cdot \log_{10}(2018) \approx 6669.331$. Disregarding the whole number part of 6669.331 we have $10^{0.331} \approx 2.143$. Therefore the first digit of 2018^{2018} is a 2, and the answer is (B).

2. Assuming that paper currency has denominations of \$20, \$10, \$5, and \$1 and that coins have denominations of 10 cents, 5 cents, and 1 cent, how many ways can you create a total of \$20 in paper currency and 18 cents in coins?

   (A) 20 (B) 24 (C) 36 (D) 48 $\boxed{\text{(E) } 60}$

   Solution. For paper currency, we have the ten possibilities of (20), $(10, 10)$, $(10, 5, 5)$, $(10, 5, 1, ..., 1)$, $(10, 1, ..., 1)$, $(5, 5, 5, 5)$, $(5, 5, 5, 1, ..., 1)$, $(5, 5, 1, ..., 1)$, $(5, 1, ..., 1))$, and $(1, ..., 1)$; for coins, we have the six possibilities of $(10, 5, 1, 1, 1)$, $(10, 1, ..., 1)$, $(5, 5, 5, 1, 1, 1)$, $(5, 5, 1, ..., 1)$, $(5, 1, ..., 1)$, and $(1, ..., 1)$. Therefore, there is a total of $10 \cdot 6 = 60$ ways, and the answer is (E).

3. If a parabola is modelled as $f(x) = ax^2 + bx + c$, has a vertex at $(5, -4)$, and $a + b + c = 12$, determine the value of c.

   $\boxed{\text{(A) } 21}$ (B) 22 (C) 23 (D) 24 (E) 25

   Solution. Notice that $f(1) = 12$ since $f(1) = a + b + c = 12$ is given. Since $(5, -4)$ is the vertex, we have $f(x) = a\,(x - 5)^2 - 4$. Plugging in $f(1) = 12$ we get $16a - 4 = 12$, which gives $a = 1$. Therefore $f(x) = x^2 - 10x + 21$, giving $c = 21$. Therefore the answer is (A).

4. Find the sum of the coefficients of the polynomial $P(x) = \left(1 - x^{2018}\right)^{2018}$.

$\boxed{\text{(A) } 0}$ (B) 1 (C) 2^{2018} (D) 2018^{2018} (E) $2^{2018^{2018}}$

Solution. The sum of the coefficients of a polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0$ can be expressed as $f(1) = a_n + \cdots + a_1 + a_0$. Therefore the answer is $P(1) = \left(1 - 1^{2018}\right)^{2018} = 0^{2018} = 0$. The answer is (A).

5. A surveillance drone starts from the origin $(0, 0)$ and flies East with a constant speed 10 feet per second. After every minute the drone turns 30 degrees left to its current direction. What is the distance (in feet) of the drone to the origin after 2018 minutes?

(A) 600 (B) 1200 (C) $600\sqrt{3}$ (D) $600(2 + \sqrt{3})$ $\boxed{\text{(E) } 600\sqrt{2 + \sqrt{3}}}$

Solution. The drone flies around a regular polygon of 12 sides (a dodecagon) because $360/30 = 12$. Therefore, the distance of the drone to the origin after 2018 minutes is the same as that after 2 minutes due to the fact that the remainder of 2018 divided by 12 is 2. It reduces to compute the length of the base of a isosceles triangle with angle $180° - 30° = 150°$ and two equal sides of 600 ft. By the law of cosines, we have the answer $600\sqrt{2 + \sqrt{3}}$ ft.

Therefore the answer is (E).

6. What is the integer part of $\sum_{n=1}^{100} \frac{1}{\sqrt{n} + \sqrt{n+2}}$?

$\boxed{\text{(A) } 8}$ (B) 9 (C) 16 (D) 17 (E) 18

Solution. By the formula:

$$\begin{aligned}
\frac{1}{\sqrt{n} + \sqrt{n+2}} &= \frac{1}{\sqrt{n} + \sqrt{n+2}} \left(\frac{\sqrt{n} - \sqrt{n+2}}{\sqrt{n} - \sqrt{n+2}} \right) \\
&= \frac{\sqrt{n} - \sqrt{n+2}}{n - (n+2)} \\
&= \frac{1}{2} \left(\sqrt{n+2} - \sqrt{n} \right),
\end{aligned}$$

the sum has cancellation, and reduces to:

$$\sum_{n=1}^{100} \frac{1}{2} \left(\sqrt{n+2} - \sqrt{n} \right) = \frac{1}{2} \left(\sqrt{102} + \sqrt{101} - \sqrt{2} - 1 \right)$$

The integer part of the sum is therefore 8, and the answer is (A).

7. In one semester at Acme High School, there are seven different courses to be assigned to three teachers such that one teacher teaches three courses, and each of the other two teachers teaches two courses. How many possible ways are there to assign the courses to the three teachers?

(A) 105 (B) 210 (C) 315 [(D) 630] (E) 1050

Solution. Step 1: choose a teacher to assign 3 courses (3 ways to do this). Step 2: choose 3 courses for this teacher ($\binom{7}{3} = 35$ ways to do this). Step 3: assign the remaining 4 courses to the remaining two teachers ($\binom{4}{2} = 6$ ways to do this). So the total number of choices is $3 \cdot 35 \cdot 6 = 630$, and the answer is (D).

8. Suppose that A and B are sets with $|A| = 10$ and $|B| = 7$. (Here $|A|$ denotes the number of elements of the set A.) Find the best bounds on x and y so that $x \leq |A \cup B| \leq y$.

(A) $x = 0$, $y = 17$ (B) $x = 0$, $y = 10$ [(C) $x = 10$, $y = 17$]
(D) $x = 6$, $y = 16$ (E) $x = 17$, $y = 17$

Solution. The set $A \cup B$ is as small as possible when $B \subset A$; in that case $|A \cup B| = |A| = 10$. The set $A \cup B$ is as large as possible when A and B share no elements in common; in that case $|A \cup B| = |A| + |B| = 17$. Therefore the answer is (C).

9. Let n be a natural number and suppose that $T(n)$ is defined by

$$T(1) = 1 \text{ and } T(n+1) = T(n) + n + 1.$$

Suppose that $S(n)$ is defined by $S(1) = 0$ and $S(n+1) = S(n) + n$. If $W(n) = T(n) - S(n)$, compute $W(20) - W(18)$.

(A) -2 (B) -1 [(C) 2] (D) 3 (E) 6

Solution. We have the following computation:

$$\begin{aligned} W(20) - W(18) &= T(20) - S(20) - (T(18) - S(18)) \\ &= (T(19) + 20) - (S(19) + 19) - T(18) + S(18) \\ &= T(19) - S(19) - T(18) + S(18) + 1 \\ &= (T(18) + 19) - (S(18) + 18) - T(18) + S(18) + 1 \\ &= 2. \end{aligned}$$

Therefore $W(20) - W(18) = 2$ and the answer is (C).

10. A coin is flipped 10 times. What is the probability that there are exactly five heads given that the first three flips are heads?

(A) $\frac{1}{2^7}$ (B) $\boxed{\frac{21}{2^7}}$ (C) $\frac{21}{2^{10}}$ (D) $\frac{252}{2^{10}}$ (E) $\frac{1}{4}$

Solution. This is equivalent to the question: what is the probability of getting exactly two heads in 7 flips? There are $\binom{7}{2} = \frac{7 \cdot 6}{2!} = 21$ possible outcomes of 7 flips with exactly two heads. For example $TTHTTHT$, $HTHTTTT$, $THTTHTT \ldots$ etc. Each outcome is equally likely with probability of $\frac{1}{2^7}$. Therefore the probability is $\frac{21}{2^7}$ and the answer is (B).

11. Racquetballs are sold in a 3-pack, stacked vertically, in a cylindrical container. Find the ratio of the height of the container to the circumference of the container.

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{6}{\pi}$ (D) $\boxed{\frac{3}{\pi}}$ (E) $\frac{6}{\pi^2}$

Solution. Let d be the diameter of each ball. Assuming the balls are packed tightly, the height of the container will be $3d$ and the circumference will be πd. Thus the ratio h/C will be $(3d)/(\pi d) = 3/\pi$, and so the answer is (D).

12. How many integer pairs (a, b) are solutions of the equation $a^2 + 1 = b^2 + 2018$?

(A) infinitely many (B) 20 (C) 8
$\boxed{\text{(D) } 4}$ (E) equation has no solutions

Solution. The equation is equivalent to $a^2 - b^2 = 2017$, which is equivalent to $(a + b)(a - b) = 2017$. Since 2017 is prime, the only solutions occur when:

$$\begin{aligned}
(a+b, a-b) &= (1, 2017) & (a, b) &= (1009, -1008)\\
(a+b, a-b) &= (2017, 1) & (a, b) &= (1009, 1008)\\
(a+b, a-b) &= (-1, -2017) & (a, b) &= (-1009, 1008)\\
(a+b, a-b) &= (-2017, -1) & (a, b) &= (-1009, -1008).
\end{aligned}$$

There are four solutions to the equation, and the answer is (D).

13. Three fair dice are rolled and one denotes the values that show on the top faces by X_1, X_2, and X_3, respectively. Find the probability that $X_1 > X_2 + X_3$.

(A) $\frac{1}{3}$ (B) $\frac{21}{316}$ (C) $\boxed{\frac{20}{216}}$ (D) $\frac{36}{216}$ (E) $\frac{51}{216}$

Solution. First we will compute each of the probabilities $P(X_1 > X_2 + j)$ for $j = 1, 2, \ldots, 6$. However, $P(X_1 > X_2 + 5) = 0$ and $P(X_1 > X_2 + 6) = 0$, so we will only be considering $j = 1, 2, 3, 4$. We have:

$$\begin{aligned} P(X_1 > X_2 + j) &= P(X_2 = 1)P(X_1 > 1 + j) + \cdots + P(X_2 = 4)P(X_1 > 4 + j) \\ &= \left(\frac{1}{6}\right)\left(\frac{5-j}{6}\right) + \cdots + \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) \\ &= \left(\frac{1}{36}\right)((5-j) + (4-j) + \cdots + 2 + 1). \end{aligned}$$

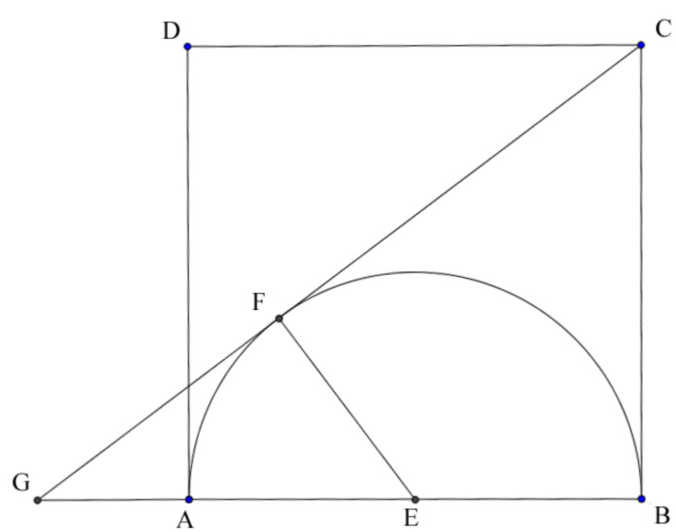
Therefore, $P(X_1 > X_2 + 1) = \left(\frac{1}{36}\right)(4 + 3 + 2 + 1) = \frac{10}{36}$, $P(X_1 > X_2 + 2) = \left(\frac{1}{36}\right)(3 + 2 + 1) = \frac{6}{36}$, and so on: $P(X_1 > X_2 + 3) = \frac{3}{36}$, $P(X_1 > X_2 + 4) = \frac{1}{36}$.

Now we calculate the probability $P(X_1 > X_2 + X_3)$.

$$\begin{aligned} P(X_1 > X_2 + X_3) &= P(X_3 = 1)P(X_1 > X_2 + 1) + \cdots + P(X_3 = 4)P(X_1 > X_2 + 4) \\ &= \left(\frac{1}{6}\right)\left(\frac{10}{36}\right) + \left(\frac{1}{6}\right)\left(\frac{6}{36}\right) + \left(\frac{1}{6}\right)\left(\frac{3}{36}\right) + \left(\frac{1}{6}\right)\left(\frac{1}{36}\right) \\ &= \left(\frac{1}{216}\right)(10 + 6 + 3 + 1) \\ &= \frac{20}{216}. \end{aligned}$$

Therefore the answer is (C).

14. A square $ABCD$ has side length 2, E is the midpoint of AB, and G is a point on the ray $\overrightarrow{BA}$ such that CG is tangent to the circle centered at E with radius AE. Let the point of tangency be F. Find the area of $\triangle EFG$.



(A) 1 $\quad$ **(B)** $\boxed{\frac{2}{3}}$ $\quad$ (C) $\frac{4}{3}$ $\quad$ (D) $\frac{5}{6}$ $\quad$ (E) $\frac{5}{3}$

Solution. First solution: Let $\angle BCE = \alpha$, then $\angle BCF = 2\alpha$. Because $\tan\alpha = |BE|/|BC| = 1/2$, we have $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = 4/3$. Since $\triangle EFG \sim \triangle CBG$, we have $|GF|/|FE| = |GB|/|CB| = \tan\angle BCF = 4/3$, and then $|GF| = (4/3)|FE| = (4/3)(1) = 4/3$. So the area of $\triangle EFG = (1/2)(4/3)(1) = 2/3$.

Second solution: Let $|GF| = x$, $|GA| = y$, then the relation $|GF|^2 = |GA|\cdot|GB|$ (due to the tangent-secant theorem for circles) gives $x^2 = y(y+2)$, and the relation $|GF|/|FE| = |GB|/|BC|$ (due to $\triangle EFG \sim \triangle CBG$) gives $x/1 = (y+2)/2$. We obtain $x = 4/3$ by solving the system of equations, and the area of $\triangle EFG = (1/2)(4/3)(1) = 2/3$. Therefore, the answer is (B).

15. Given a sequence $\{a_k\}$ with $a_0 = (1+\sqrt{5})/2$, $a_{k+1} = a_k^2 - 2$ $(k \geq 0)$, then a_{2018} is ...

$\boxed{\text{(A) } \frac{-1-\sqrt{5}}{2}}$ $\quad$ (B) $\frac{\sqrt{5}-1}{2}$ $\quad$ (C) $\quad$ (D) 24 $\quad$ (E) 25

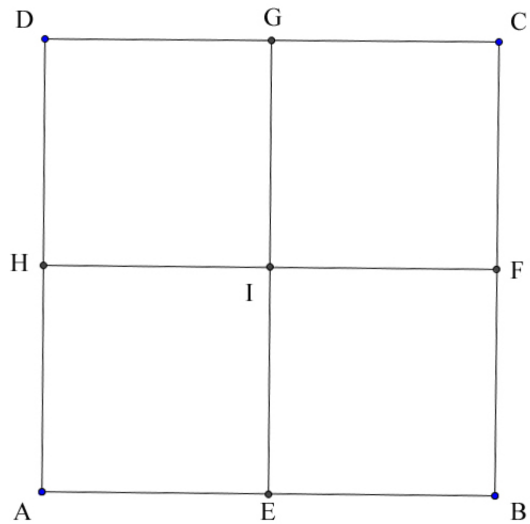
Solution. We compute that $a_1 = a_0^2 - 2 = (\sqrt{5}-1)/2$, $a_2 = a_1^2 - 2 = (-1-\sqrt{5})/2$, and then $a_{2n-1} = a_1$, $a_{2n} = a_2$ for all integers $n \geq 1$. So $a_{2018} = a_2 = (-1-\sqrt{5})/2$, and the answer is (A).

16. If $f(x) = \sin x$, for $x \in [\pi/2, 3\pi/2]$, find the value of $f^{-1}(\cos 4)$.

(A) 4 $\quad$ (B) $4-\pi$ $\quad$ (C) $\frac{\pi}{2} - 4$ $\quad$ (D) $\frac{3\pi}{2} - 4$ $\quad$ $\boxed{\text{(E) } \frac{5\pi}{2} - 4}$

Solution. We need to find a value x such that $x \in [\pi/2, 3\pi/2]$ and $\sin x = \cos 4$. Since $\cos 4 = \sin(\frac{\pi}{2} - 4) = \sin(\frac{\pi}{2} - 4 + 2\pi) = \sin(\frac{5\pi}{2} - 4)$, and $\frac{5\pi}{2} - 4 \in [\pi/2, 3\pi/2]$, this is the desired value, and the answer is (E).

17. A square $ABCD$ is divided into four small squares as shown below. How many ways are there to travel from A to C along the sides of the squares without repetitions of any sides (repetitions of vertices are allowed)?



(A) 6 (B) 10 (C) 12 (D) 14 (E) 16

Solution. The ways starting with AH are: AHDGC, AHDGIFC, AHDGIEBFC, AHIGC, AHIFC, AHIFBEIGC, AHIEBFC, AHIEBFIGC. By symmetry, there are another eight distinct ways starting with AE. The total is 16 and the answer is (E).

18. Let a, b and c be positive real numbers such that $\log_3(a) = .5$, $\log_3(b) = .2$, and $\log_3(c) = .3$. Compute the exact value of $\log_3(a^2b^5 - abc + 3)$.

(A) 2 (B) 3 (C) 4 (D) 9.8 (E) 10

Solution. We have the following:

$$\begin{array}{lcl} \log_3(a) = 0.5 & & a = \sqrt{3} = \sqrt[10]{3^5} \\ \log_3(b) = 0.2 & \longleftrightarrow & b = \sqrt[5]{3} = \sqrt[10]{3^2} \\ \log_3(c) = 0.3 & & c = \sqrt[10]{3^3}. \end{array}$$

Then we may compute $\log_3(a^2b^5 - abc + 3) = \log_3(3 \cdot 3 - \sqrt[10]{3^5 \cdot 3^2 \cdot 3^3} + 3) = \log_3(9) = 2$, and so the answer is (A).

19. Find the coefficient of the x^9 term in the expansion of

$$(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)(x-8)(x-9)(x-10).$$

(A) 55 (B) 45 (C) −45 (D) −50 (E) −55

Solution. There are ten terms in the expansion of the form ax^9, they are:

$$\begin{aligned}-x^9 - 2x^9 - 3x^9 - \cdots - 10x^9 &= -(1+2+3+\cdots+10)x^9 \\ &= -\left(\frac{10\cdot 11}{2}\right)x^9 \\ &= -55x^9.\end{aligned}$$

Therefore, the answer is (E).

20. Consider all permutations of the symbols a, b, c, d and e. Let A be a set of these permutations that satisfy the following axioms: a is in an odd numbered position in the permutation, b appears before c, and d must be in a position immediately after e's position. Which of the following permutations are elements of the set A?

I: *cabed* **II:** *abdec* **III:** *bceda* **IV:** *acbed*

(A) **I** only (B) **II** only [(C) **III** only] (D) **II** and **III** (E) **IV** only

Solution. We consider each permutation and whether it is in the set A. We assume that the five positions are numbered one, two, three, four, and five; rather than zero, one, two, three, and four.

- **I:** *cabed* (a is in position two, therefore *cabed* is not in A)
- **II:** *abdec* (d is not immediately after e, therefore *abdec* is not in A)
- [**III:** *bceda* (the axioms are satisfied, therefore *bceda* is in A)]
- **IV:** *acbed* (b appears after c, therefore *acbed* is not in A)

Therefore the answer is (C).

21. Through how many points with integer coordinates does the circle $x^2 + y^2 = 2018$ pass?

(A) 0 [(B) 8] (C) 16 (D) 24 (E) 32

Solution. First, x and y are of the same parity because 2018 is even. As 2018 is not a multiple of 4, x and y can not be both even, so they must be both odd. For an odd number, the unit digit of its square can only be 1, 5, or 9, so the unit digit of x^2 and y^2 must be both 9. It follows that the absolute values of x and y can only be two numbers of the set $\{3, 7, 13, 17, 23, 27, 33, 37, 43\}$. Square these numbers, and observe the sums of pairs (or examine the difference between 2018 and these squares); we see that $(43, 13)$, $(43, -13)$, $(-43, 13)$, $(-43, -13)$, $(13, 43)$, $(13, -43)$, $(-13, 43)$, $(-13, -43)$ are the eight solutions.

Therefore the answer is (B).

22. In the World Series, the best of 7 games series is finished when one team has won 4 games. Assuming that each team has a 50% chance of winning each game, determine the probability that Game 6 will be the last game of the series.

$\boxed{\text{(A) } \frac{5}{16}}$ (B) $\frac{5}{32}$ (C) $\frac{5}{64}$ (D) $\frac{2}{64}$ (E) $\frac{1}{64}$

Solution. Call the teams A and B. We will assume that A wins the series in six games, calculate the probability, and then double this probability to account for the case where B wins the series in six games. If A wins the series in six games, then A won three games and lost two in the first five games of the series; finally winning game 6. The probability of this happening would be:

$$\binom{5}{3} \cdot \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right) = \frac{5}{32}.$$

The probability of the series ending in six games is therefore $\frac{5}{16}$, and the answer is (A).

23. Consider the equation $x^2 - 12x + \sqrt{1100} = 0$, where all the numbers are written in base 3. (As a reminder, 12 in base 3, also written as 12_3, is the number $1 \cdot 3^1 + 2 = 5$ in base 10. Similarly, 200 in base 3 equals $2 \cdot 3^2 + 0 + 0 = 18$ in base 10, and so on.) Write in base 3 the solutions of this equation.

(A) 1, 2 $\boxed{\text{(B) 2, 10}}$ (C) 10, 11
(D) 12, $\sqrt{1100}$ (E) equation has complex roots

Solution. The equation in our usual base 10 representation reads:

$$\begin{aligned} x^2 - (1 \cdot 3 + 2)x + \sqrt{1 \cdot 3^3 + 1 \cdot 3^2 + 0 \cdot 3 + 0} &= x^2 - 5x + 6 \\ &= (x-2)(x-3). \end{aligned}$$

The two solutions are 2 and 3, written in base 3 the two solutions are 2 and 10. Therefore the answer is (B).

24. How many counting numbers n are such that the fraction $\dfrac{2n+3}{3n+1}$ can be further simplified?

(A) 1 (B) 3 (C) 21 (D) 2018 $\boxed{\text{(E) infinitely many}}$

Solution. We see that for any $n = 7k + 2$ ($k \in \mathbb{Z}$), the fraction can be reduced. This implies that there are infinitely many counting numbers n such that the fraction can be reduced, and so the answer is (E).

Remark: It appears that, when the numerator and the denominator of the fraction are replaced by two general polynomials $p(x)$, $q(x)$ with integer coefficients, we have the general result that $p(n)/q(n)$ either can be reduced for infinitely many n, or it cannot be reduced for any integer n. In fact, if $k = \gcd(p(n_0), q(n_0)) > 1$ for some $n_0 \in \mathbb{Z}$, then k is also a common factor of $p(nk + n_0)$ and $q(nk + n_0)$ for all $n \in \mathbb{Z}$ because $k|(p(nk + n_0) - p(n_0))$ and $k|(q(nk + n_0) - q(n_0))$.

25. The sign function, $\text{sgn}(x)$, is defined to equal 1 if x is positive, 0 if $x = 0$, and -1 if x is negative. Consider the function $f(x) = x \cdot \text{sgn}(x - 1)$. Find the value $f^{30}(0.3)$. (Here f^n denotes the composition of functions: $f^n(x) = f(f(\ldots f(f(x))\ldots))$, n times.)

(A) 0.3^{30} (B) -0.3 **[(C) 0.3]** (D) 0.7 (E) 0.7^{30}

Solution. We calculate:

$$\begin{aligned} f(0.3) &= (0.3) \cdot \text{sgn}(-0.7) = -0.3 \\ f^2(0.3) &= f(-0.3) = (-0.3) \cdot \text{sgn}(-1.3) = 0.3 \\ f^3(0.3) &= f(0.3) = -0.3 \\ &\vdots \end{aligned}$$

One can see that $f^n(0.3) = 0.3$ if n is even, and $f^n(0.3) = -0.3$ if n is odd. Therefore $f^{30}(0.3) = 0.3$ and the answer is (C).

26. If a and b are the two roots of the equation $x^2 + 2018x - 1 = 0$, then the value of $a^{2018} + ab^{-2017} + a^{2017}b + b^{-2016}$ is ...

[(A) 0] (B) 2017 (C) 2018 (D) 2018^{2018} (E) -2018

Solution. The product of the two roots must be the constant coefficient, so $ab = -1$. Therefore $a^{2017} + b^{-2017} = a^{2017} - a^{2017} = 0$. If we rewrite the expression in the problem as $(a+b)(a^{2017} + b^{-2017})$, we can conclude that it is zero. Therefore the answer is (A).

27. How many 6-digit positive even integers are there such that the six digits are distinct numbers in the set $\{1, 2, 3, 4, 5, 6\}$ and that all the absolute differences of adjacent digits are no more than 4?

(A) 216 **[(B) 240]** (C) 288 (D) 480 (E) 600

Solution. We need to count the number of permutations of these six digits such that the last digit is even and 1 is not adjacent to 6. There are $6!/2 = 360$ permutations with the last digit even. For those permutations with the last digit even and 1 adjacent to 6, there some cases to consider:

i. 16 is at the end (there are $4! = 24$ choices),

ii. 16 occurs in the number with a 2 at the end (there are $4 \cdot 3! = 24$ choices),

iii. 16 occurs in the number with a 4 at the end (there are $4 \cdot 3! = 24$ choices),

iv. 61 occurs in the number with a 2 at the end (there are $4 \cdot 3! = 24$ choices),

v. 61 occurs in the number with a 4 at the end (there are $4 \cdot 3! = 24$ choices).

Putting everything together, there are $360 - 5 \cdot 24 = 240$ such numbers, and therefore the answer is (B).

28. Let L be a line that goes through the point $(1, 3)$. Suppose that L has the y-intercept at $(0, y_0)$ and the x-intercept at $(x_0, 0)$. Assume that $x_0, y_0 > 0$. Suppose that the area of the triangle bounded by L, the x-axis and y-axis is 6 square units. Find values of x_0 and y_0.

(A) $x_0 = 4/3$, $y_0 = 9$ (B) $x_0 = 9$, $y_0 = 4/3$ (C) $x_0 = 4$, $y_0 = 4$
(D) $x_0 = 2$, $y_0 = 6$ (E) $x_0 = 6$, $y_0 = 2$

Solution. The area of the triangle with vertices $(0, 0)$, $(x_0, 0)$, and $(0, y_0)$ is 6 units^2, therefore $x_0y_0/2 = 6$ or $x_0y_0 = 12$. The line L has slope $-y_0/x_0$ and y-intercept y_0, so the equation of L is given by $y = -\frac{y_0}{x_0}x + y_0$. Since $\frac{1}{x_0} = \frac{y_0}{12}$ the equation of L can be rewritten $y = -y_0 \cdot \left(\frac{y_0}{12}\right) x + y_0 = -\frac{y_0^2}{12}x + y_0$. The line L passes through $(1, 3)$, so we have:

$$3 = -\frac{y_0^2}{12}(1) + y_0 \iff y_0^2 - 12y_0 + 36 = 0 \iff (y_0 - 6)^2 = 0 \iff y_0 = 6.$$

Therefore $(x_0, y_0) = (2, 6)$, and the answer is (D).

29. For each subset A of the natural numbers, define $f(A) = \sum_{a \in A} 10^{-a}$. In which situation given below are the sets A and B disjoint?

I: $f(A) = .101010$ and $f(B) = .011101$.

II: $f(A) = .1010$ and $f(B) = .0101$.

III: $f(A) = .0001$ and $f(B) = .1001$.

IV: $f(A) = .1000$ and $f(B) = .0111$.

(A) **I** only (B) **II** only (C) **IV** only [(D) **II** and **IV**] (E) **II** and **III**

Solution. We determine the sets A and B in each case.

- **I:** $A = \{1, 3, 5\}$, $B = \{2, 3, 4, 5, 7\}$ (A and B are not disjoint)
- [**II:** $A = \{1, 3\}$, $B = \{2, 4\}$ (A and B are disjoint)]
- **III:** $A = \{4\}$, $B = \{1, 4\}$ (A and B are not disjoint)
- [**IV:** $A = \{1\}$, $B = \{2, 3, 4\}$ (A and B are disjoint)]

Therefore the answer is (D).

30. In how many ways can we make a list of three integers (a, b, c) where $0 \leq a, b, c \leq 9$ so that $a + b + c$ is odd and a is even?

(A) 100 (B) 125 (C) 150 (D) 200 [(E) 250]

Solution. Since a is even and $a + b + c$ is odd, exactly one of b, c must be odd. There are $5 \cdot 5 \cdot 5 = 125$ choices for the list (a, b, c) with a, b even and c odd; and there are $5 \cdot 5 \cdot 5 = 125$ choices for the list (a, b, c), with a, c even and b odd. There are 250 total, and the answer is (E).

31. A circle has points A, B, C, and D placed equidistant around the circumference of the circle so that $\overline{AC}$ and $\overline{BD}$ form diameters, intersecting at center O. A circle is drawn tangent to radius $\overline{OA}$, radius $\overline{OB}$, and arc $\overset{\frown}{AB}$. Determine the ratio of the area of the larger circle to the smaller circle.

(A) $2 + \sqrt{3}$ (B) $3\sqrt{2} + 4$ [(C) $3 + 2\sqrt{2}$] (D) 4 (E) $2 + 3\sqrt{2}$

Solution. Let R be the radius of the original circle and r be the radius of the smaller circle. One can use a 45-45-90 triangle to justify that the distance between the centers of the two circles is $r\sqrt{2}$. Therefore $R = r + r\sqrt{2} = (1 + \sqrt{2})r$. The desired ratio of the two areas is therefore:

$$\begin{aligned} \frac{\pi R^2}{\pi r^2} &= \frac{\pi(1 + \sqrt{2})^2 r^2}{\pi r^2} \\ &= (1 + \sqrt{2})^2 \\ &= 3 + 2\sqrt{2}. \end{aligned}$$

Therefore the answer is (C).

32. Ron Swanson is paddling a canoe down a river. The current of the river is flowing at a constant speed. Paddling with the current, Mr. Swanson can go 50% faster than paddling against the flow. Determine the ratio of Mr. Swanson's paddling speed to the flow of the current.

(A) 2.5 (B) 3 (C) 4 **[(D) 5]** (E) 7.5

Solution. Let p be Mr. Swanson's paddling speed, c the current speed. We have that $p+c = 1.5\cdot(p-c)$. Dividing both sides by c we obtain $p/c+1 = 1.5(p/c)-1.5$. Solving for p/c gives $p/c = 5$, and the answer is (D).

Assuming the Ron Swanson in this question is based on the character of Ronald Ulysses Swanson from *Parks and Recreation*, let us conclude this solution with one of his quotations:

"There's only one thing I hate more than lying: skim milk. Which is water that is lying about being milk."

33. Consider the function $f(x) = x^3 + 5x^2 - 30x - 20$. What is the sum of the reciprocals of the three roots of the function?

(A) -5 **[(B) $-\frac{3}{2}$]** (C) 1 (D) $-\frac{2}{9}$ (E) $-\frac{1}{4}$

Solution. Consider the equation:

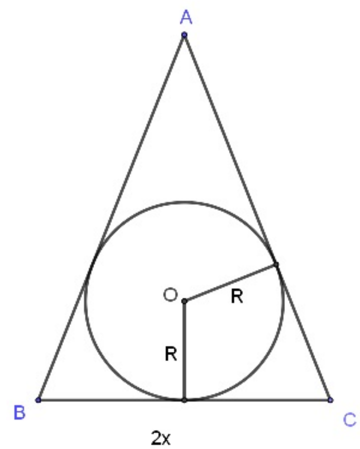
$$\begin{aligned} f(x) &= x^3 + 5x^2 - 30x - 20 \\ &= (x - r_1)(x - r_2)(x - r_3) \\ &= x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3. \end{aligned}$$

We are trying to find the sum of the reciprocals of the three roots, therefore:

$$\begin{aligned} \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} &= \frac{r_2r_3 + r_1r_3 + r_1r_2}{r_1r_2r_3} \\ &= \frac{-30}{20} \\ &= -\frac{3}{2}. \end{aligned}$$

Therefore the answer is (B).

34. An isosceles triangle $\triangle ABC$ with base $2x$ is circumscribed to a circle of radius R. Express in terms of x and R the perimeter of the triangle.



(A) $4x + 2\sqrt{x^2 + R^2}$ (B) $\boxed{2x + \dfrac{2x(x^2 + R^2)}{x^2 - R^2}}$ (C) $2x + 4\sqrt{3}R$

(D) $2x + \dfrac{2x}{R}\sqrt{x^2 + R^2}$ (E) $\dfrac{2x}{R}(x^2 + R^2 + R)$

Solution. Let D be the point of tangency of the circle and the line AC. Let AD have length y and DC we know has length x. Let θ be the angle $\angle BCO$. We calculate:

$$\begin{aligned}\frac{x}{x+y} = \cos(2\theta) &= 2\cos^2(\theta) - 1 \\ &= 2\left(\frac{x}{\sqrt{x^2 + R^2}}\right)^2 - 1 \\ &= \frac{2x^2 - (R^2 + x^2)}{x^2 + R^2} \\ &= \frac{x^2 - R^2}{x^2 + R^2}.\end{aligned}$$

Solving for y we obtain

$$y = x \cdot \frac{x^2 + R^2}{x^2 - R^2} - x,$$

and finally the perimeter is given by

$$\begin{aligned} P = 4x + 2y &= 4x + 2x \cdot \frac{x^2 + R^2}{x^2 - R^2} - 2x \\ &= 2x + 2x \cdot \frac{x^2 + R^2}{x^2 - R^2}. \end{aligned}$$

Therefore the answer is (B).

35. If α and β are the complex roots of the equation $x^2 + x + 1 = 0$, calculate

$$(1+\alpha)^{2018} + (1+\beta)^{2018} + (\alpha+\beta)^{2018}.$$

$\boxed{\text{(A) } 0}$ (B) $\alpha + \beta$ (C) $\alpha^2 + \beta^2$ (D) $(\alpha+\beta)^{-1}$ (E) $2\alpha\beta$

Solution. We will make use of the following properties:

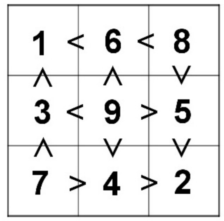
- $1 + \alpha = -\alpha^2$ (α is a root of $x^2 + x + 1$)
- $1 + \beta = -\beta^2$ (β is a root of $x^2 + x + 1$)
- $\alpha + \beta = -1$ (follows from $x^2 + x + 1 = (x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$)
- $\alpha^3 = 1$ (follows from $x^3 - 1 = (x - 1)(x^2 + x + 1)$)
- $\beta^3 = 1$ (same reason as above)

We may then calculate:

$$\begin{aligned} (1+\alpha)^{2018} + (1+\beta)^{2018} + (\alpha+\beta)^{2018} &= (-\alpha^2)^{2018} + (-\beta^2)^{2018} + (-1)^{2018} \\ &= \alpha^{4036} + \beta^{4036} + 1 \\ &= (\alpha^3)^{1345}\alpha + (\beta^3)^{1345}\beta + 1 \\ &= \alpha + \beta + 1 \\ &= -1 + 1 \\ &= 0. \end{aligned}$$

Therefore the answer is (A).

36. The numbers from 1 to 9 are arranged in a square grid as shown below. One also indicates on the edges of the adjacent little squares the inequality between the numbers in the respective squares. If one tries to rearrange the numbers while preserving the inequalities, what numbers can be placed in the middle of the grid?



(A) 9 only (B) 9 and 8 **(C) $9, 8, 7$** (D) $9, 8, 7, 6$ (E) $9, 8, 7, 6, 5$

Solution. With some experimentation, one can find valid grids with an 8 or 7 at the center; some examples are illustrated below.

1	5	9	1	6	7	1	5	9	2	5	8
3	8	6	4	8	5	3	7	6	4	7	6
7	4	2	9	3	2	8	4	2	9	3	1

However, there is no valid grid with a 6 (or lower) at the center, so the answer is (C).

37. Given three circles O_1, O_2, O_3 with radii 2, 3, 10, respectively, O_1 and O_2 are tangent externally at a point A, O_2 and O_3 are tangent externally at a point B, and O_3 and O_1 are tangent externally at a point C, then the length of AB is ...

(A) $2\sqrt{2}$ (B) $3\sqrt{2}$ (C) $\dfrac{144}{13}$ **(D) $\dfrac{12}{\sqrt{13}}$** (E) 3

Solution. We first find $\cos(\angle AO_2B) = 5/13$ by the law of cosines in the triangle $O_1O_2O_3$ (where O_1, O_2, O_3 are the centers of the circles, and $|O_1O_2| = 2+3 = 5$, $|O_2O_3| = 3+10 = 13$, $|O_3O_1| = 10+2 = 12$). Alternatively, one may observe that $\cos(\angle AO_2B) = 5/13$, as $\triangle O_1O_2O_3$ is a right triangle since $5^2 + 12^2 = 13^2$). Then we proceed to find $|AB|$ by the law of cosines in $\triangle AO_2B$:

$$\begin{aligned} |AB|^2 &= 3^2 + 3^2 - 2(3)(3)\cos(\angle A)_2B) \\ &= 18 - 18\left(\frac{5}{13}\right) \\ &= 18\left(\frac{8}{13}\right) \\ &= \frac{2^4 \cdot 3^2}{13}, \end{aligned}$$

therefore $|AB| = \frac{12}{\sqrt{13}}$ and the answer is (D).

38. Let P, Q, R be points on the sides $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ of $\triangle ABC$ with $\lambda = \frac{AP}{PB} = \frac{BQ}{QC} = \frac{CR}{RA} < 1$. If the ratio of the area of $\triangle PQR$ to the area of $\triangle ABC$ is $\frac{7}{12}$, then the value of λ is ...

(A) $\dfrac{1}{3}$ (B) $\dfrac{1}{4}$ **(C) $\dfrac{1}{5}$** (D) $\dfrac{1}{6}$ (E) $\dfrac{11-6\sqrt{2}}{7}$

Solution. Without loss of generality, we assume that the area of $\triangle ABC$ is 1. Notice that the common ratio of the sides $\frac{AP}{AB} = \frac{BQ}{BC} = \frac{CR}{CA}$ would necessarily be equal to the ratio $\frac{\lambda}{\lambda+1}$. Then the area of $\triangle APR$ can be computed $\frac{\lambda}{1+\lambda} \cdot \frac{1}{1+\lambda} = \frac{\lambda}{(1+\lambda)^2}$, since base and height are scaled by factors of $\frac{\lambda}{1+\lambda}$ and $\frac{1}{1+\lambda}$ respectively. Similarly, $\triangle BQP$ and $\triangle CRQ$ also have area $\frac{\lambda}{(1+\lambda)^2}$. So the area of $\triangle PQR$ is $1 - \frac{3\lambda}{(1+\lambda)^2}$ which gives the equation $1 - \frac{3\lambda}{(1+\lambda)^2} = \frac{7}{12}$. Solving this equation we get:

$$\begin{aligned} \frac{5}{12} &= \frac{3\lambda}{(1+\lambda)^2} \\ 5\lambda^2 - 26\lambda + 5 &= 0 \\ (5\lambda - 1)(\lambda - 5) &= 0. \end{aligned}$$

Therefore $\lambda = \frac{1}{5}$ (we ignore the solution $\lambda = 5$ since $\lambda < 1$). The answer is (C).

39. Let θ be the acute angle between the lines $1009x - 2018y = 2017$ and $673x + 2019y = 2018$. What is the value of $\tan\theta$?

(A) $\dfrac{1}{4}$ (B) $\dfrac{1}{3}$ (C) $\dfrac{1}{2}$ (D) $\dfrac{5}{7}$ **(E) 1**

Solution. Let the (signed) angles of the lines to the positive side of the x-axis be α and β, respectively. Then $\tan\alpha$, $\tan\beta$ are the slopes of the lines, which are $1/2$ and $-1/3$. We have $\tan\theta = |\tan(\alpha-\beta)| = \frac{|\tan\alpha - \tan\beta|}{|1+\tan\alpha\tan\beta|} = 1$, so the answer is (E).

40. Let a and b be integers and suppose that eight does not divide $a^2 - b^2$. Which of the following statements must be true?

(A) a and b are both odd (B) a and b are both even

(C) one of a or b is even, and the other is odd **(D) a and b are not both odd**

(E) a and b are both even, or both odd

Solution. Since 8 does not divide $5 = 3^2 - 2^2$, the statements (A), (B), and (E) are not necessarily true. Also, 8 does not divide $12 = 4^2 - 2^2$, and so (C) is not necessarily true. This leaves only (D) remaining.

Now we argue that if $a^2 - b^2$ is not divisible by 8, then a and b are not both odd. Consider the square of an odd number. There are four possibilities:

- the odd number is of the form $8k+1$, then the square is of the form $8r+1$, since $(8k+1)^2 = 64k^2 + 16k + 1 = 8(8k^2 + 2k) + 1$ (let $r = 8k^2 + 2k$)
- the odd number is of the form $8k+3$, then the square is of the form $8r+1$, since $(8k+3)^2 = 64k^2 + 48k + 9 = 8(8k^2 + 6k + 1) + 1$ (let $r = 8k^2 + 6k + 1$)
- the odd number is of the form $8k+5$, then the square is of the form $8r+1$, since $(8k+5)^2 = 64k^2 + 80k + 25 = 8(8k^2 + 10k + 3) + 1$ (let $r = 8k^2 + 10k + 3$)
- the odd number is of the form $8k+7$, then the square is of the form $8r+1$, since $(8k+7)^2 = 64k^2 + 112k + 49 = 8(8k^2 + 14k + 6) + 1$ (let $r = 8k^2 + 14k + 6$).

Therefore, if a and b are both odd, then a^2 and b^2 are of the form $8r+1$ and $8k+1$ respectively. Then $a^2 - b^2 = (8r+1) - (8k+1) = 8(r-k)$, which is divisible by 8. Conclusion: *if $a^2 - b^2$ is not divisible by 8, then a and b are not both odd.* The answer is (D).

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