THE SIXTY-SECOND ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

Part II

Solutions

1.i. When $x = 1$, E coincides with B, and $|AG| = |BG| = 1 = |AB|$; so $\triangle ABG$ is equilateral and $\angle GBA = \angle GAB = 60°$. Consequently, $\angle GBC = \angle GAD = 30°$, and $\angle FAB = 30°$ by symmetry; then $\angle FAG = 90° - 30° - 30° = 30°$. It follows that Region I and Region II are congruent, and therefore $r(1) = 1$.

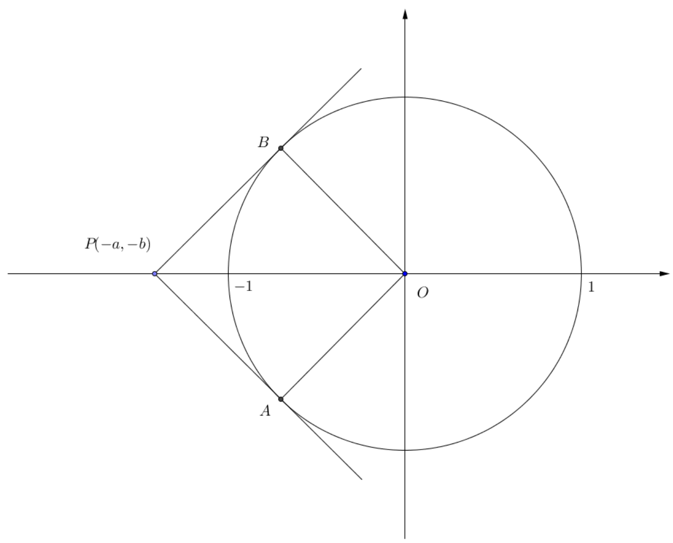
1.ii. We show that $r(x) = \frac{1}{x^2}$ $(0 < x \leq 1)$. In fact, we have $\angle GAB = \angle AGB$ because $\triangle GAB$ is isosceles with base GA; hence $\angle GAB = \frac{1}{2}(180° - \angle GBA) = \frac{1}{2}(180° - (90° - \angle GBC)) = 45° + \frac{1}{2}\angle GBC$. Then $\angle FAB = \angle GAD = 90° - \angle GAB = 45° - \frac{1}{2}\angle GBC$. It derives that $\angle GAF = 90° - \angle FAB - \angle GAD = 90° - 2(45° - \frac{1}{2}\angle GBC) = \angle GBC$. Therefore the chords GC and GF correspond to equal central angles in circle Γ_1 and Γ_3, respectively. It follows that the two regions are similar, and the ratio $r(x)$ of their areas equals to the square of $|GC|/|GF|$. Noticing that $|GC|/|GF|$ equals to the ratio of the radii of circle Γ_1 and circle Γ_3, we obtain $r(x) = \left(\frac{1}{x}\right)^2 = \frac{1}{x^2}$.

2.i. Because $\deg(v_3) = 3$, v_3 knows v_1, v_2, and v_3, and v_1 only knows v_3 because $\deg(v_1) = 1$. It follows that v_2 knows v_3 and v_4 due to $\deg(v_2) = 2$. Then v_4 knows v_2 and v_3 only, and hence $\deg(v_4) = 2$.

2.ii. We use the mathematical induction on k with $n = 4k$. The base case $k = 1$ is true by the previous question. Suppose that the result holds for some $k \geq 1$, and consider a party $V = \{v_1, \ldots, v_{4k+4}\}$ with $n = 4(k+1)$ and $\deg(v_i) = i$ for all $i = 1, \ldots, 4k+3$. Note that v_{4k+3} knows everyone else in the party, and then v_1 only knows v_{4k+3}; so if we drop out v_{4k+3} and v_1 from the party, the degrees for $v_2, \ldots, v_{4k+1}, v_{4k+2}$ becomes $1, 2, \ldots, 4k, 4k+1$ in the party $V' = \{v_2, \ldots, v_{4k+2}, v_{4k}\}$. Now v_{4k+2} knows everyone else in V', and therefore v_2 only knows v_{4k+2} in V'. We further drop out v_2 and v_{4k+2} from V', then the degrees for $v_3, \ldots, v_{4k+1}$ become $1, 2, \ldots, 4k-1$ in $V'' = \{v_3, \ldots, v_{4k+1}, v_{4k}\}$. Now apply the induction assumption on V'', we derive that v_{4k} knows $4k/2 = 2k$ people in V''; so $\deg(v_{4k}) = 2k + 2 = n/2$ in the original party V because v_{4k} also knows v_{4k+2} and v_{4k+3}; hence the conclusion holds for $n = 4(k+1)$. By mathematical induction, the result holds for any $n = 4k$.

3.i. Any ordered pair (a, b) in the regions $\{(a,b) \mid a > 1, b \geq 2a + \sqrt{5}\}$ or $\{(a,b) \mid a < -1, b \leq 2a - \sqrt{5}\}$ works. But since we only need an example, we could choose those that are easier to justify. For example, pick $(a, b) = (2, 7)$; then $0 < a + \cos x \leq 3$, $b + \sin x \geq 6$, and $f(x) \geq 6/3 = 2$.

3.ii. **First solution.** We regard $f(x)$ as the slope of the line passing the point $P(-a,-b)$ and a moving point $(\cos x, \sin x)$ on the unit circle. Since $a > 1$, P lies outside the unit circle to the left side of the vertical line $x = -1$. Let PA, PB be the tangent lines from P to the unit circle with points of tangency A and B, respectively; as illustrated in the graph below. Then the range of the function $f(x)$ is exactly the set of numbers between the slopes (inclusive) of the lines PA and PB. Therefore PA has slope -1, and PB has slope 1. It derives that $OAPB$ is a square with side length 1, and P must be on the x-axis with $|OP| = \sqrt{|PA|^2 + |OA|^2} = \sqrt{2}$. Therefore the coordinate of P is $(-\sqrt{2}, 0)$, or $a = \sqrt{2}$, $b = 0$.



Second solution. Let $y = \frac{b+\sin x}{a+\cos x}$, we have that $y \cos x - \sin x = b - ay$ or

$$\sqrt{1+y^2}\sin(x+\theta) = b - ay \tag{1}$$

with θ some angle depending on y. It follows that $|b - ay| \leq \sqrt{1+y^2}$ or, equivalently,

$$(a^2-1)y^2 - 2aby + b^2 - 1 \leq 0 \tag{2}$$

We see that if y is given and (2) holds, then we can always find x to satisfy (1), and conversely, if (1) holds for some x, then (2) holds. It follows that the solution set to (2) is the range of the function $f(x)$, which is $[-1, 1]$ as given in the problem. Since $a^2 - 1 > 0$ due to $a > 1$, the solution set to (2) is the closed interval bounded by the two roots of $(a^2-1)y^2 - 2aby + b^2 - 1 = 0$. Therefore the two roots are ± 1; so $2ab/(a^2-1) = 1 + (-1) = 0$, and $(b^2-1)/(a^2-1) = (1)(-1) = -1$. We then get $a = \sqrt{2}$ and $b = 0$.

4.i. If $120 = a^2 - b^2$ for some positive odd numbers a and b, then $a > b$ and $120 = (a-b)(a+b)$. Because $a-b$, $a+b$ are both even, and $\frac{a-b}{2} + \frac{a+b}{2} = a$ is odd, one of $\frac{a-b}{2}$ and $\frac{a+b}{2}$ must be odd. So one of $a-b$ and $a+b$ is of the form $4k+2$ for some nonnegative integer k. We then find the qualified factorizations of 120 as $120 = 2 \cdot 60 = 6 \cdot 20 = 10 \cdot 12 = 4 \cdot 30$, and obtain correspondingly that $120 = 31^2 - 29^2 = 13^2 - 7^2 = 11^2 - 1^2 = 17^2 - 13^2$. So $f(120) = 4$.

4.ii. Let $x = 8k$ for some positive integer k. We first show that $f(x)$ *is the number of positive odd factors of* x. If $8k = (2r+1)^2 - (2s+1)^2 = 4(r+s+1)(r-s)$ for some $r > s \geq 0$, then $(r+s+1)(r-s) = 2k$. Note that $r+s+1 \geq r+1 > r \geq r-s$, and $r+s+1$, $r-s$ are of different parity, so one of them must be an odd factor of $2k$. Conversely, if we have $ab = 2k$ with $a > b$ and one of a, b odd (the other then is even), then $r+s+1 = a$, $r-s = b$ gives $r = (a+b-1)/2$, $s = (a-b-1)/2$, which are both nonnegative integers that satisfy $8k = (2r+1)^2 - (2s+1)^2$. It follows that $f(x)$ is the number of positive odd factors of $2k$, which is the same as the number of positive odd factors of x. So $f(x) = 1$ if x has no odd prime factors, and $f(x) = (\alpha_1 + 1) \cdots (\alpha_l + 1)$ if $x = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_l^{\alpha_l}$ with $\alpha_0 \geq 3$, $\alpha_1, \ldots, \alpha_l \geq 1$, $l > 0$ and $p_1, \ldots, p_l$ are distinct positive odd primes.

Since $8 = 1 \cdot 8 = 2 \cdot 4 = 2 \cdot 2 \cdot 2$, x has the form $2^{\alpha_0} p_1^7$, $2^{\alpha_0} p_1 p_2^3$, or $2^{\alpha_0} p_1 p_2 p_3$ if $f(x) = 8$, where $\alpha_0 \geq 3$, p_1, p_2, p_3 are distinct positive odd primes. To make x small, we of course put $\alpha_0 = 3$, and choose p_1, p_2, p_3 as small as possible. Comparing 3^7, $3 \cdot 5^3$, $3^3 \cdot 5$, and $3 \cdot 5 \cdot 7$, we see that $3 \cdot 5 \cdot 7 = 105$ is the minimum among the four numbers, and therefore the smallest possible x for which $f(x) = 8$ is $2^3 \cdot 3 \cdot 5 \cdot 7 = 840$.

4.iii. By the above formula, we see that $f(2^3 \cdot p^{\alpha-1}) = \alpha$ for any odd positive prime p and $\alpha = 1, 2, \ldots$, which gives the result.

5. **First proof.** For any positive integer k, write $k = 2^{\alpha_k} \beta_k$ such that β_k is an odd integer, and α_k is an nonnegative integer; clearly this representation is unique by the prime factorization theorem.

Sufficiency. Suppose that $n = 2^m$ for some integer $m > 0$, and let $1 \leq r \leq n-1$. Then $\alpha_k < m$ for all $1 \leq k \leq r$. It follows that $\frac{n-k}{k} = \frac{2^m - 2^{\alpha_k}\beta_k}{2^{\alpha_k}\beta_k} = \frac{2^{m-\alpha_k} - \beta_k}{\beta_k}$, with the last fraction having both the numerator and the denominator odd ($1 \leq k \leq r-1$), and $\frac{n}{r} = \frac{2^m}{2^{\alpha_r}\beta_r} = \frac{2^{m-\alpha_r}}{\beta_r}$, with the last fraction having an even numerator and an odd denominator. It follows that

$$C_{n,r} = \frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n}{r} \times \frac{n-1}{1} \times \cdots \times \frac{n-r+1}{r-1}$$

can be written as a fraction having an even numerator and an odd denominator. Since $C(n,r)$ is an integer, it then must be even.

Necessity. Suppose that $n \geq 2$ is not a power of 2. We need to show that there is r with $1 \leq r \leq n-1$ such that $C_{n,r}$ is odd. Let $m \geq 1$ be the largest integer such that $2^m < n$, and let $r = n - 2^m$. Then $1 \leq r < 2^m \leq n-1$. We claim that $C_{n,r}$ is odd. In fact, for all $1 \leq k \leq r$, we have $\alpha_k < m$, and therefore $\frac{2^m+k}{k} = \frac{2^m+2^{\alpha_k}\beta_k}{2^{\alpha_k}\beta_k} = \frac{2^{m-\alpha_k}+\beta_k}{\beta_k}$ with the last fraction having both the numerator and the denominator odd. So

$$C_{n,r} = C_{2^m+r,r} = \frac{(2^m+1)\cdots(2^m+r)}{r!} = \frac{2^m+1}{1} \times \cdots \times \frac{2^m+r}{r}$$

can be written as a fraction having both the numerator and the denominator odd. Since $C_{n,r}$ is an integer, it then must be odd.

Second proof. For two polynomials $f(x) = a_n x^n + \cdots + a_0$ and $g(x) = b_n x^n + \cdots + b_0$ with integer coefficients, we say that $f(x) \equiv g(x) \pmod 2$ if $a_i \equiv b_i \pmod 2$ for all $0 \leq i \leq n$. Then the problem becomes that $(1+x)^n \equiv 1 + x^n \pmod 2$ $(n \geq 2)$ iff n is a power of 2. In the following we use the fact that $f_1(x) \equiv g_1(x) \pmod 2$ and $f_2(x) \equiv g_2(x) \pmod 2$ imply $f_1(x)f_2(x) \equiv g_1(x)g_2(x) \pmod 2$.

Sufficiency. We use the mathematical induction on m when $n = 2^m$. When $m = 1$, $2^m = 2$, and $(1+x)^2 = 1 + 2x + x^2 \equiv 1 + x^2 \pmod 2$, so the result holds. Now suppose that $(1+x)^{2^m} \equiv 1 + x^{2^m} \pmod 2$ for some integer $m \geq 1$, then $(1+x)^{2^{m+1}} = [(1+x)^{2^m}]^2 \equiv [1 + x^{2^m}]^2 \equiv 1 + 2x^{2^m} + x^{2^{m+1}} \equiv 1 + x^{2^{m+1}} \pmod 2$; so the result holds for $n = 2^{m+1}$. By the mathematical induction, the result holds whenever $n \geq 2$ is a power of 2.

Necessity. Suppose that $n \geq 2$ is not a power of 2. We need to show that there is r with $1 \leq r \leq n-1$ such that $C_{n,r}$ is odd. Let $m \geq 1$ be the largest integer such that $2^m < n$, and let $r = n - 2^m$. Then $1 \leq r < 2^m$. We derive by the sufficiency that

$$\begin{aligned}
(1+x)^n &= (1+x)^{2^m}(1+x)^r \\
&\equiv (1+x^{2^m})(\sum_{k=0}^{r} C_{r,k}x^k) \\
&\equiv \sum_{k=0}^{r} C_{r,k}x^k + \sum_{k=0}^{r} C_{r,k}x^{k+2^m} \pmod 2.
\end{aligned}$$

Since $r < 2^m$, we see that the coefficient of x^{2^m} in the last polynomial above is $C_{r,0} = 1$; so $C_{n,2^m} \equiv 1 \pmod 2$, which completes the proof.

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