MMPC STUDENT CODE

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THE SIXTY-SECOND ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America

Part II

Wednesday, December 5, 2018

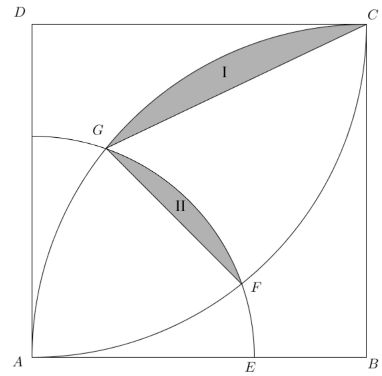
INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. Indicate whether we may use your e-mail address to contact you.**BUT DO NOT ACTUALLY WRITE YOUR NAME OR E-MAIL ADDRESS ON THIS BOOKLET.**
2. Part II consists of problems and proofs. You will be allowed 100 minutes (1 hour and 40 minutes) for the five questions. To receive full credit for a problem, you are expected to justify your answer.
3. You are not expected to solve all problems completely. Look over all the problems and work first on those that interest you the most. If you are unable to solve a particular problem, partial credit might be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.
4. Each problem is on a separate page. If possible, you should show all of your work on that page. If there is not enough room, you may continue your solution on the blank page at the end of the booklet (page 7) or on additional paper inserted into the examination booklet. Be certain to **check the appropriate box** to report where your continuation occurs. On the continuation page clearly write the **problem number**. If you use additional paper for your answer, check the appropriate box and write your **identification number** and the **problem number** in the upper right-hand corner of each additional sheet.
5. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate an idea of how to solve the problem. On the other hand, a carefully stated generalization may actually be easier to prove than the problem as stated.
6. The competition rules prohibit you from asking questions of anyone during the examination. The use of notes, reference material, computation aids, or any other aid is likewise prohibited. Please note that **calculators are not allowed** on this exam. When the supervisor announces that the 100 minutes are over, please cease work immediately and insert all significant extra paper into the test booklet. Please do not return scratch paper containing routine numerical calculations.
7. You may now open the test booklet.

#	1	2	3	4	5	Total
Score						

1. Let $ABCD$ be a square with side length 1, Γ_1 be a circle centered at B with radius 1, Γ_2 be a circle centered at D with radius 1, E be a point on the segment AB with $|AE| = x$ $(0 < x \leq 1)$, and Γ_3 be a circle centered at A with radius $|AE|$. Γ_3 intersects Γ_1 and Γ_2 inside the square at G and F, respectively. Let region I be the region bounded by the segment GC and the minor arc $\widehat{GC}$ of Γ_1, and region II be the region bounded by the segment FG and the minor arc $\widehat{FG}$ of Γ_3, as illustrated in the graph below.



Let $r(x)$ be the ratio of the area of region I to the area of region II.

(i) Find $r(1)$; justify your answer.

(ii) Find an explicit formula of $r(x)$ in terms of x $(0 < x \leq 1)$; justify your answer.

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2. We call a *party* any set of people V. If $v_1 \in V$ knows $v_2 \in V$ in a party, we always assume that v_2 also knows v_1. For a person $v \in V$ in some party, the *degree* of v, denoted by $\deg(v)$, is the number of people v knows in the party.

   (i) Suppose that a party has four people with $V = \{v_1, v_2, v_3, v_4\}$, and that $\deg(v_i) = i$ for $i = 1, 2, 3$; show that $\deg(v_4) = 2$.

   (ii) Suppose that a party is attended by $n = 4k$ $(k \geq 1)$ people with $V = \{v_1, v_2, \ldots, v_{4k}\}$, and that $\deg(v_i) = i$ for $1 \leq i \leq n - 1$; show that $\deg(v_n) = \frac{n}{2}$.

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3. Let a, b be two real number parameters and consider the function $f(x) = \dfrac{b + \sin x}{a + \cos x}$.

   (i) Find an example of (a, b) such that $f(x) \geq 2$ for all real numbers x. Justify your answer.

   (ii) If $a > 1$ and the range of the function $f(x)$ (when x varies over the set of all real numbers) is $[-1, 1]$, find the values of a and b; justify your answer.

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4. Let f be the function that assigns to each positive multiple x of 8 the number of ways in which x can be written as a difference of squares of positive odd integers. (For example, $f(8) = 1$, because $8 = 3^2 - 1^2$, and $f(24) = 2$, because $24 = 5^2 - 1^2 = 7^2 - 5^2$.)

   (a) Determine with proof the value of $f(120)$.

   (b) Determine with proof the smallest value x for which $f(x) = 8$.

   (c) Show that the range of this function is the set of all positive integers.

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5. Consider the binomial coefficients $C_{n,r} = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$, for $n \geq 2$. Prove that $C_{n,r}$ are even, for *all* $1 \leq r \leq n-1$, if and only if $n = 2^m$, for some counting number m.

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(Continued Solutions)

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America

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