# THE SIXTY-SECOND ANNUAL
# MICHIGAN MATHEMATICS PRIZE COMPETITION

Sponsored by

The Michigan Section of the Mathematical Association of America

## Part I

Tuesday, October 9, 2018

## INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Carefully read and follow the instructions printed on the answer sheet. Check to ensure that your five-digit MMPC code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor asks you to stop, please quit working immediately and turn in your answer sheet.

3. Consider the problems and responses carefully. You may work out ideas on scratch paper before selecting a response.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number of correct answers. You are advised to guess an answer in those cases where you cannot determine an answer.

5. For each of the questions, five different possible responses are provided. Choose the correct answer and completely fill in the corresponding bubble on your answer sheet.

6. Any scientific or graphing calculator is permitted on Part I. Unacceptable machines include computers, PDAs, pocket organizers, cell phones, and similar devices. All problems will be solvable with no more technology than a scientific calculator. The Exam Committee makes every effort to structure the test to minimize the advantage of a more powerful calculator. No other devices are permitted.

7. No one is permitted to explain to you the meaning of any question. Do not ask anyone to violate the rules of the competition. If you have questions concerning the instructions, ask them now.

8. You may open the test booklet and begin.

**1.** When $2018^{2018}$ is expanded in base ten, what is the sum of the first digit and the last digit?

(A) 5 (B) 6 (C) 7 (D) 8 (E) 10

**2.** Assuming that paper currency has denominations of \$20, \$10, \$5, and \$1 and that coins have denominations of 10 cents, 5 cents, and 1 cent, how many ways can you create a total of \$20 in paper currency and 18 cents in coins?

(A) 20 (B) 24 (C) 36 (D) 48 (E) 60

**3.** If a parabola is modelled as $f(x) = ax^2 + bx + c$, has a vertex at $(5, -4)$, and $a + b + c = 12$, determine the value of $c$.

(A) 21 (B) 22 (C) 23 (D) 24 (E) 25

**4.** Find the sum of the coefficients of the polynomial $P(x) = \left(1 - x^{2018}\right)^{2018}$.

(A) 0 (B) 1 (C) $2^{2018}$ (D) $2018^{2018}$ (E) $2^{2018^{2018}}$

**5.** A surveillance drone starts from the origin $(0, 0)$ and flies East with a constant speed 10 feet per second. After every minute the drone turns 30 degrees left to its current direction. What is the distance (in feet) of the drone to the origin after 2018 minutes?

(A) 600 (B) 1200 (C) $600\sqrt{3}$ (D) $600(2 + \sqrt{3})$ (E) $600\sqrt{2 + \sqrt{3}}$

**6.** What is the integer part of $\sum_{n=1}^{100} \frac{1}{\sqrt{n}+\sqrt{n+2}}$?

(A) 8 (B) 9 (C) 16 (D) 17 (E) 18

**7.** In one semester at Acme High School, there are seven different courses to be assigned to three teachers such that one teacher teaches three courses, and each of the other two teachers teaches two courses. How many possible ways are there to assign the courses to the three teachers?

(A) 105 (B) 210 (C) 315 (D) 630 (E) 1050

**8.** Suppose that $A$ and $B$ are sets with $|A| = 10$ and $|B| = 7$. (Here $|A|$ denotes the number of elements of the set $A$.) Find the best bounds on $x$ and $y$ so that $x \leq |A \cup B| \leq y$.

(A) $x = 0,\ y = 17$ (B) $x = 0,\ y = 10$ (C) $x = 10,\ y = 17$
(D) $x = 6,\ y = 16$ (E) $x = 17,\ y = 17$

**9.** Let $n$ be a natural number and suppose that $T(n)$ is defined by

$$T(1) = 1 \text{ and } T(n+1) = T(n) + n + 1.$$

Suppose that $S(n)$ is defined by $S(1) = 0$ and $S(n+1) = S(n) + n$. If $W(n) = T(n) - S(n)$, compute $W(20) - W(18)$.

(A) -2 (B) -1 (C) 2 (D) 3 (E) 6

**10.** A coin is flipped 10 times. What is the probability that there are exactly five heads given that the first three flips are heads?

(A) $\frac{1}{2^7}$ (B) $\frac{21}{2^7}$ (C) $\frac{21}{2^{10}}$ (D) $\frac{252}{2^{10}}$ (E) $\frac{1}{4}$

**11.** Racquetballs are sold in a 3-pack, stacked vertically, in a cylindrical container. Find the ratio of the height of the container to the circumference of the container.

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{6}{\pi}$ (D) $\frac{3}{\pi}$ (E) $\frac{6}{\pi^2}$

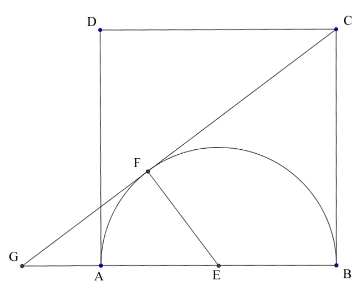
**12.** How many integer pairs $(a, b)$ are solutions of the equation $a^2 + 1 = b^2 + 2018$?

(A) infinitely many (B) 20 (C) 8 (D) 4
(E) equation has no solutions

**13.** Three fair dice are rolled and one denotes the values that show on the top faces by $X_1$, $X_2$, and $X_3$, respectively. Find the probability that $X_1 > X_2 + X_3$.

(A) $\frac{1}{3}$ (B) $\frac{21}{36}$ (C) $\frac{20}{216}$ (D) $\frac{36}{216}$ (E) $\frac{51}{216}$

**14.** A square $ABCD$ has side length 2, $E$ is the midpoint of $AB$, and $G$ is a point on the ray $\overrightarrow{BA}$ such that $CG$ is tangent to the circle centered at $E$ with radius $AE$. Let the point of tangency be $F$. Find the area of $\triangle EFG$.




(A) 1 (B) $\frac{2}{3}$ (C) $\frac{4}{3}$ (D) $\frac{5}{6}$ (E) $\frac{5}{3}$

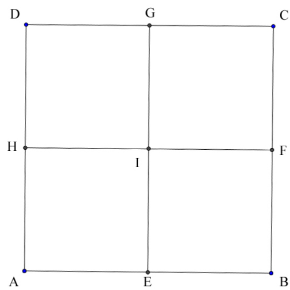
**15.** Given a sequence $\{a_k\}$ with $a_0 = (1+\sqrt{5})/2$, $a_{k+1} = a_k^2 - 2$ $(k \geq 0)$, then $a_{2018}$ is ...

(A) $\frac{-1-\sqrt{5}}{2}$ (B) $\frac{\sqrt{5}-1}{2}$ (C) $\frac{1-\sqrt{5}}{2}$ (D) $\left(\frac{1-\sqrt{5}}{2}\right)^{2018}$ (E) $\left(\frac{1+\sqrt{5}}{2}\right)^{2018}$

**16.** If $f(x) = \sin x$, for $x \in [\pi/2, 3\pi/2]$, find the value of $f^{-1}(\cos 4)$.

(A) 4 (B) $4 - \pi$ (C) $\frac{\pi}{2} - 4$ (D) $\frac{3\pi}{2} - 4$ (E) $\frac{5\pi}{2} - 4$

**17.** A square $ABCD$ is divided into four small squares as shown below. How many ways are there to travel from $A$ to $C$ along the sides of the squares without repetitions of any sides (repetitions of vertices are allowed)?




(A) 6 (B) 10 (C) 12 (D) 14 (E) 16

**18.** Let $a$, $b$ and $c$ be positive real numbers such that $\log_3(a) = 0.5$, $\log_3(b) = 0.2$, and $\log_3(c) = 0.3$. Compute the exact value of $\log_3(a^2b^5 - abc + 3)$.

(A) 2 (B) 3 (C) 4 (D) 9.8 (E) 10

**19.** Find the coefficient of the $x^9$ term in the expansion of

$$(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)(x-8)(x-9)(x-10).$$

(A) 55 (B) 45 (C) -45 (D) -50 (E) -55

**20.** Consider all permutations of the symbols $a$, $b$, $c$, $d$ and $e$. Let $A$ be a set of these permutations that satisfy the following axioms: $a$ is in an odd numbered position in the permutation, $b$ appears before $c$, and $d$ must be in a position immediately after $e$'s position. Which of the following permutations are elements of the set $A$?

**I:** *cabed*; **II:** *abdec*; **III:** *bceda*; **IV:** *acbed*.

(A) **I** only (B) **II** only (C) **III** only (D) **II** and **III** (E) **IV** only

**21.** Through how many points with integer coordinates does the circle $x^2 + y^2 = 2018$ pass?

(A) 0 (B) 8 (C) 16 (D) 24 (E) 32

**22.** In the World Series, the best of 7 games series is finished when one team has won 4 games. Assuming that each team has a 50% chance of winning each game, determine the probability that Game 6 will be the last game of the series.

(A) $\frac{5}{16}$ (B) $\frac{5}{32}$ (C) $\frac{5}{64}$ (D) $\frac{2}{64}$ (E) $\frac{1}{64}$

**23.** Consider the equation $x^2 - 12x + \sqrt{1100} = 0$, where all the numbers are written in base 3. (As a reminder, 12 in base 3, also written as $12_3$, is the number $1 \cdot 3^1 + 2 = 5$ in base 10. Similarly, 200 in base 3 equals $2 \cdot 3^2 + 0 \cdot 3^1 + 0 = 18$ in base 10, and so on.) Write in base 3 the solutions of this equation.

(A) 1,2 (B) 2, 10 (C) 10, 11 (D) 12, $\sqrt{1100}$
(E) equation has complex roots

**24.** How many counting numbers $n$ are such that the fraction $\dfrac{2n+3}{3n+1}$ can be further simplified?

(A) 1 (B) 3 (C) 21 (D) 2018 (E) infinitely many

**25.** The sign function, $\text{sgn}(x)$, is defined to equal 1 if $x$ is positive, 0 if $x = 0$, and $-1$ if $x$ is negative. Consider the function $f(x) = x \cdot \text{sgn}(x-1)$. Find the value $f^{30}(0.3)$. (Here $f^n$ denotes the composition of functions: $f^n(x) = f(f(\ldots f(f(x))\ldots))$, $n$ times.)

(A) $0.3^{30}$ (B) $-0.3$ (C) 0.3 (D) 0.7 (E) $0.7^{30}$

**26.** If $a$ and $b$ are the two roots of the equation $x^2 + 2018x - 1 = 0$, then the value of $a^{2018} + ab^{-2017} + a^{2017}b + b^{-2016}$ is ...

(A) 0 (B) 2017 (C) 2018 (D) $2018^{2018}$ (E) $-2018$

**27.** How many 6-digit positive even integers are there such that the six digits are distinct numbers in the set $\{1, 2, 3, 4, 5, 6\}$ and that all the absolute differences of adjacent digits are no more than 4?

(A) 216 (B) 240 (C) 288 (D) 480 (E) 600

**28.** Let $L$ be a line that goes through the point $(1, 3)$. Suppose that $L$ has the $y$-intercept at $(0, y_0)$ and the $x$-intercept at $(x_0, 0)$. Assume that $x_0, y_0 > 0$. Suppose that the area of the triangle bounded by $L$, the $x$-axis and $y$-axis is 6 square units. Find values of $x_0$ and $y_0$.

(A) $x_0 = 4/3, y_0 = 9$ (B) $x_0 = 9, y_0 = 4/3$ (C) $x_0 = 4, y_0 = 4$
(D) $x_0 = 2, y_0 = 6$ (E) $x_0 = 6, y_0 = 2$

**29.** For each subset $A$ of the natural numbers, define $f(A) = \sum_{a \in A} 10^{-a}$. In which situation given below are the sets $A$ and $B$ disjoint?

**I:** $f(A) = .101010$ and $f(B) = .011101$.
**II:** $f(A) = .1010$ and $f(B) = .0101$.
**III:** $f(A) = .0001$ and $f(B) = .1001$.
**IV:** $f(A) = .1000$ and $f(B) = .0111$.

(A) **I** only (B) **II** only (C) **IV** only (D) **II** and **IV** (E) **II** and **III**

**30.** In how many ways can we make a list of three integers $(a, b, c)$, where $0 \le a, b, c \le 9$, so that $a + b + c$ is odd and $a$ is even?

(A) 100 (B) 125 (C) 150 (D) 200 (E) 250

**31.** A circle has points $A$, $B$, $C$, and $D$ placed equidistant around the circumference of the circle so that $\overline{AC}$ and $\overline{BD}$ form diameters, intersecting at center $O$. A circle is drawn tangent to radius $\overline{OA}$, radius $\overline{OB}$, and arc $\overset{\frown}{AB}$. Determine the ratio of the area of the larger circle to the smaller circle.

(A) $2 + \sqrt{3}$ (B) $3\sqrt{2} + 4$ (C) $3 + 2\sqrt{2}$ (D) 4 (E) $2 + 3\sqrt{2}$

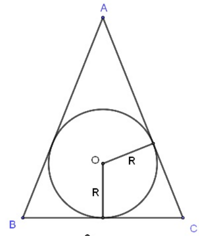
**32.** Ron Swanson is paddling a canoe down a river. The current of the river is flowing at a constant speed. Paddling with the current, Mr. Swanson can go 50% faster than paddling against the flow. Determine the ratio of Mr. Swanson's paddling speed to the flow of the current.

(A) 2.5 (B) 3 (C) 4 (D) 5 (E) 7.5

**33.** Consider the function $f(x) = x^3 + 5x^2 - 30x - 20$. What is the sum of the reciprocals of the three roots of the function?

(A) -5 (B) -$\frac{3}{2}$ (C) -1 (D) -$\frac{2}{9}$ (E) -$\frac{1}{4}$

**34.** An isosceles triangle $\triangle ABC$ with base $2x$ is circumscribed to a circle of radius $R$. Express in terms of $x$ and $R$ the perimeter of the triangle.



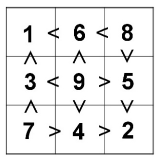

(A) $4x + 2\sqrt{x^2 + R^2}$ (B) $2x + \frac{2x(x^2+R^2)}{x^2-R^2}$ (C) $2x + 4\sqrt{3}R$
(D) $2x + \frac{2x}{R}\sqrt{x^2 + R^2}$ (E) $\frac{2x}{R}(x^2 + R^2 + R)$

**35.** If $\alpha$ and $\beta$ are the complex roots of the equation $x^2+x+1=0$, calculate

$$(1+\alpha)^{2018}+(1+\beta)^{2018}+(\alpha+\beta)^{2018}.$$

(A) $0$ (B) $\alpha+\beta$ (C) $\alpha^2+\beta^2$ (D) $(\alpha+\beta)^{-1}$ (E) $2\alpha\beta$

**36.** The numbers from 1 to 9 are arranged in a square grid as shown below. Indicated on the edges of the adjacent little squares are the inequalities between the numbers in the respective squares. If one tries to rearrange the numbers while preserving the inequalities, what numbers can be placed in the middle of the grid?




(A) 9 only (B) 9 and 8 (C) 9, 8, 7 (D) 9, 8, 7, 6 (E) 9, 8, 7, 6, 5

**37.** Given three circles $O_1$, $O_2$, $O_3$ with radii 2, 3, 10, respectively, $O_1$ and $O_2$ are tangent externally at a point $A$, $O_2$ and $O_3$ are tangent externally at a point $B$, and $O_3$ and $O_1$ are tangent externally at a point $C$, then the length of $AB$ is ...

(A) $2\sqrt{2}$ (B) $3\sqrt{2}$ (C) $\frac{144}{13}$ (D) $\frac{12}{\sqrt{13}}$ (E) $3$

**38.** Let $P$, $Q$, $R$ be points on the sides $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ of $\triangle ABC$ with $\lambda = \frac{AP}{PB} = \frac{BQ}{QC} = \frac{CR}{RA} < 1$. If the ratio of the area of $\triangle PQR$ to the area of $\triangle ABC$ is $\frac{7}{12}$, then the value of $\lambda$ is ...

(A) $\frac{1}{3}$ (B) $\frac{1}{4}$ (C) $\frac{1}{5}$ (D) $\frac{1}{6}$ (E) $\frac{11-6\sqrt{2}}{7}$

**39.** Let $\theta$ be the acute angle between the lines $1009x - 2018y = 2017$ and $673x + 2019y = 2018$. What is the value of $\tan\theta$?

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{5}{7}$ (E) $1$

**40.** Let $a$ and $b$ be integers and suppose that eight does not divide $a^2-b^2$. Which of the following statements must be true?

(A) $a$ and $b$ are both odd (B) $a$ and $b$ are both even
(C) one of $a$ or $b$ is even, and the other is odd (D) $a$ and $b$ are not both odd
(E) $a$ and $b$ are both even, or both odd

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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