MMPC 2017 Part I
Solutions

1. (D) In the first four exams, Oscar scores a total of $83 \cdot 4 = 332$ points. In the first five exams, Oscar scores a total of $84 \cdot 5 = 420$ points. The difference is 88.

2. (B) Suppose there are now 100 animals. Then 60 are cats and 40 are dogs. If there were 10% fewer cats and 35% more dogs, there would be 54 cats and also 54 dogs. The new percentage would be 50.

3. (B) The middle number is the average of all the numbers, which is 1. There are 1008 numbers after the middle number, so the last number is 1009.

4. (C) The sum of all the angles in the shaded triangles is $7 \cdot 180°$. These angles include the angles we care about and two sets of exterior angles for the inner heptagon. The exterior angles of a polygon sum to $360° = 2 \cdot 180°$, so the other angles sum to $(7 - 2 \cdot 2) \cdot 180° = 540°$.

5. (A)

$$\begin{aligned} |(2017)(201720172017) - (20172017)(20172017)| &= 2017^2|100010001 - (10001)^2| \\ &= 2017^2|100010001 - 100020001| = 2017^2 \cdot 10000 \end{aligned}$$

Since $40000000000 < 2017^2 \cdot 10000 < 90000000000$, this is an 11-digit number.

6. (C) First, note that $\alpha + \beta + \gamma = \pi$. Then $\gamma = 2\alpha + \beta > \alpha + \beta = \pi - \gamma$, so $\gamma > \pi/2$. This triangle has an obtuse angle.

7. (E) A fraction represents a terminating decimal if it can be written in the form $\frac{m}{10^k}$, i.e. if it can be written with a denominator whose only prime factors are 2 and/or 5. Thus the only condition is that n is a multiple of 7. There are 99 multiples of 7 in this interval. (The decimals are $.01, .02, .03, \ldots, .99$.)

8. (A) The common value of each type of coin must be a multiple of 50 cents; suppose it is $50m$ cents. Then there are $50m$ pennies, $10m$ nickels, $5m$ dimes, and $2m$ quarters, for a total of $67m$ coins. The only choice which is a multiple of 67 is 335.

9. (D) Let M be the intersection of AC and BD, and let $a = AM, b = BM, c = CM.d = DM$. Then

$$AB^2 + CD^2 = a^2 + b^2 + c^2 + d^2 = a^2 + d^2 + b^2 + c^2 = AD^2 + BC^2.$$

Therefore $9^2 + 12^2 = AD^2 + 5^2$, so $AD = \sqrt{200} = 10\sqrt{2}$.

10. (B) Solving $y = (x-1)^2$ for x gives $x = 1 \pm \sqrt{y}$. Because we are concerned with values of $x < 0$, we must have the negative sign.

11. (D) Since $\frac{\sqrt[3]{n}}{\sqrt[5]{n}} = n^{2/15} = (\sqrt[15]{n})^2$, it is necessary and sufficient for a number to be square. There are $\lfloor\sqrt{2017}\rfloor = 44$ such numbers.

12. (C) First, notice that $(f \circ f)(x) = -(-x + 2017) + 2017 = x$, so $f \circ f$ is the identity.

$$\underbrace{(f \circ f \circ \cdots \circ f)}_{2017 \text{ times}}(x) = f(x) = 2017 - x.$$

The only solution of $2017 - x = x$ is $x = 2017/2$.

13. (C) Triangles ADF and EBF are similar with length ratio 1 to 2. Their heights (from F) sum to 1, so the height of the shaded triangle is 2/3. The base is 2. The area is $(1/2)(2/3)2 = 2/3$.

14. (A) These three points all lie on $z = x + y + 1$. The only choice which also satisfies this equation is $(-1, -1, -1)$.

15. (A)

$$\begin{aligned} \log_2(9 - 2^x) &> 3 - x \\ 9 - 2^x &> 2^{3-x} \\ 9 - 2^x &> 8 \cdot 2^{-x} \\ (2^x)^2 - 9(2^x) + 8 &< 0 \\ (2^x - 1)(2^x - 8) &< 0 \end{aligned}$$

This is equivalent to $1 < 2^x < 8$, or $0 < x < 3$.

16. (C) By considering the factors of 10, we see that the number must either be $\pm 10, \pm 1, \pm 1, \pm 1$, or $\pm 5, \pm 2, \pm 1, \pm 1$. The first case is impossible since the three ± 1 could not be distinct. In the other case, the ± 1 must have opposite signs, so the other two numbers must also have opposite signs. The possibilities are $5, -2, 1, -1$ and $-5, 2, 1, -1$. Each of these can be ordered in 4! ways for a total of 48 quadruples.

17. (B) Let the side lengths be x, y, z. Then we are given $2(xy + yz + zx) = 28$ and $x^2 + y^2 + z^2 = 6^2$. Summing these and factoring gives $(x + y + z)^2 = 64$, so $x + y + z = 8$. The prism has four edges with each length, so the sum of all the lengths is $4(x + y + z) = 32$.

18. (D) Let A be the point $(6, 2)$ and let B be its reflection. Because AB is perpendicular to $y = 2x$, it has slope $-1/2$. So AB is the line $(y - 2) = \frac{-1}{2}(x - 6)$ This line intersects $y = 2x$ at $(2, 4)$. Because $(2, 4)$ is the midpoint of AB, B must be $(-2, 6)$.

19. (B) Let M be midpoint of AB. Let O be the center of the circle. Let r be the radius of the circle. If B has coordinates $(0, 0)$, then M has coordinates $(0, 1/2)$ and O has coordinates $(1 - r, 1 - r)$. Appliy the distance formula to MO.

$$(1/2 + r)^2 = (1/2 - r)^2 + (1 - r)^2$$

This simplifies to $r^2 - 4r + 1 = 0$. The quadratic formula gives $r = 2 \pm \sqrt{3}$, and since we must have $r < 1/2$, the minus sign must be chosen. (The plus sign corresponds to the situation where the circle is large enough to contain the square, and is tangent to the “other half” of the semicircle.

20. (D) Since the midpoint between the given points is not on the positive axis, they are not opposite vertices. The other vertices must be at (a, b) and $(a + 3, b - 3)$ for some a, b. The diagonals intersect at $(\frac{a+5}{2}, \frac{b-2}{2})$, so $b = 2$ and $a > -5$. Now the vertices of the parallelogram are, in cyclic order, $(2, 1), (5, -2), (a + 3, -1), (a, 2)$. By the Shoelace formula, the area is $|3a - 3|$. Since this must equal 12, $a = 5$ or $a = -3$. The possible other vertices are $(5, 2), (8, -1), (-3, 2), (0, -1)$. Only the first of these is an option.

21. (A) Modulo 3, the terms repeat $1, 1, 2, 2, 0, 2, 1, 0, \ldots$. Since $2017 = 8 \cdot 252 + 1$, there are 252 full periods (each with 2 multiples of three), the remaining number is not a multiple of three. There are 504 multiples of 3.

22. (E) Let the line be $y = 3x + b$. Then the x-coordinates of A, B are the solutions of $3x + b = x^2 - 3x + 6$, which are the roots of the quadratic equation $x^2 - 6x + (6 - b) = 0$. The sum of the roots of this equation is 6, so the average of the x-coordinates is 3.

23. (D) Let A' be the midpoint of BC; let C' be the midpoint of AB. Then O is the intersection of AA' and CC', and $x = OC'$ is the distance we seek. Notice that triangles $AA'B$ and $AC'O$ are similar 3-4-5 triangles. This gives the proportionality $\frac{AA'}{A'B} = \frac{AC'}{OC'}$. We can solve $4/3 = (5/2)/x$ to find $x = 15/8$.

24. (B) First notice that the given condition does not change if we replace x by $-x$, or if we replace y by $-y$, or if we interchange the roles of x and y. So it will be enough to find the graph in the region $x > y > 0$, and then reflect that graph over $y = x$ and the axes.

If $x > y > 0$, the equation is simply $(x+y)-(x-y) = 1$, i.e.$y = 1/2$. This gives the ray $y = 1/2, x > 1/2$. Reflecting this in the lines listed earlier gives a total of eight rays.

25. (A) Divide the hexagon into six equilateral triangles. The original triangle has 6 times the area of each of these triangles, so it has $\sqrt{6}$ times the side length of the hexagon. If s is the side length of the hexagon, then the hexagon has perimeter $6s$ while the triangle has perimeter $3\sqrt{6}s$. The ratio is $6/(3\sqrt{6}) = \sqrt{6}/3$.

26. (C) A typical term has the following form: $(-1)^k\binom{14}{k}2^{k/3}3^{(14-k)/5}$. To make k a multiple of 3 and $14-k$ a multiple of 5, we take $k = 9$, giving the term $-1\binom{14}{9}2^3 3^1 = -24\binom{14}{5}$.

27. (D) We know the following.

- a, b, c are multiples of 4.
- a, c are are multiples of 3, but b is not.
- a, b are multiples of 25, and c is a multiple of 5, but c is not a multiple of 25.
- a, b are multiples of 7, but c is not.

Thus the gcd of b, c will be a multiple of 4 and 5, but not 3, 25, or 7. Of our choices, only 220 satisfies these conditions.

28. (E) For any even number $2k$, we can write $\frac{1}{2k} = \frac{1}{3k} + \frac{1}{6k}$. We can use this repeatedly.

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6} = \frac{1}{3} + \frac{1}{9} + \frac{1}{18} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{54} = \cdots$$

Clearly we can go on like this forever.

29. (C) This sum repeats: $1 + i - 1 - i + 1 + i - 1 - i + \cdots + 1 + i - 1 - i + 1 + i$. Each group of four terms cancels, leaving just the last two terms (or the first two, if you prefer): $1 + i$.

30. (C) The given equation rearranges to $\frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = 1$. Equivalently, $\tan(\alpha - \beta) = 1$. Since α, β are acute, $-\pi/2 < \alpha - \beta < \pi/2$, so the only possible value of $\alpha - \beta$ is $45°$.

31. (E) There are $4^3 = 64$ sequences of three letters U, D, L, R. How many of these are acceptable to us? A sequence is acceptable if has either three of the same direction, or two of one direction and one of another, provided they are not opposite directions. There are 4 of the first type. There are $4 \cdot 2 \cdot 3 = 24$ of the other type (choose the direction to repeat, choose the other direction, put them in order). In total there are 28 good sequences. $28/64 = 7/16$.

32. (D) The sequence begins $c_1 = 1, c_2 = 4, c_3 = 18, \ldots$. Since $c_1 = 1 = 2! - 1$, $c_1 + c_2 = 1 + 4 = 3! - 1$, and $c_1 + c_2 + c + 3 = 1 + 4 + 18 = 4! - 1$, we are well-motivated to answer D. (For a proof, we can see inductively that $c_n = (n+1)! - n! = n(n!) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (n-1) \cdot n \cdot n$.

33. (A) This is an equilateral triangle of side 9 missing a three equilateral triangles of side 1 from each corner. (It is also a triangle of side 15 missing three triangles of side 7.)

$$A = 9^2\frac{\sqrt{3}}{4} - 3\cdot 1^2\frac{\sqrt{3}}{4} = 78\frac{\sqrt{3}}{4} = \frac{39\sqrt{3}}{2}.$$

34. (C) Since $p(x) = q_1(x)(x^2-1) + (x+3)$, we know that $p(1) = q(1)\cdot 0 + 4 = 4$. Similarly, if $p(x) = q_2(x)(x^3-1) + r(x)$, we must have $r(1) = p(1) = 4$. Only $3x+1$ has this property.

35. (D) Note that the only restriction is that a $+1$ can never be adjacent to a -1. Let u_n be the number of such n-tuples ending in -1. Let v_n be the number ending in 0. Let w_n be the number ending in 1. By symmetry $u_n = w_n$. We want $u_5 + v_5 + w_5 = 2u_5 + v_5$.

$$u_1 = v_1 = w_1 = 1$$

$$u_{n+1} = u_n + v_n$$

$$v_{n+1} = u_n + v_n + w_n = 2u_n + v_n$$

$$u_1 = 1, v_1 = 1, u_2 = 2, v_2 = 3, u_3 = 5, v_3 = 7, u_4 = 12, v_4 = 17, u_5 = 29, v_5 = 41.$$

$$29 + 41 + 29 = 99$$

36. (E) Of the 36 equally likely outcomes of Anjali and Bao rolling their dice, 21 are wins for Bao, 10 are wins for Anjali, and 5 are indecisive. There are 31 equally likely possibilities for the decisive roll. Of these, 10 are wins for Anjali.

37. (B) For large negative x, $f(x)$ approaches $-\infty$. For this function to be one-to-one, it must be increasing for large positive x. Thus we must have $m > 0$. The vertex of the parabola is at $x = m$, so in the $m > 0$ case the quadratic function is strictly increasing on $x <= 0$.

38. (C)

$$(2^x - 3)(3^x - 2)(4^x - 9)$$

This looks like it should have three solutions, $x = \log_2 3, x = \log_3 2, x = \log_4 9$, but the first and last of these are the same!

39. (E) By Vieta, the sum of the roos is $-(-4)/2 = 2$. If two of the roots sum to 1, the other root must be $2 - 1 = 1$. If 1 is a root of this polynomial, then $2 - 4 - 7 + \lambda = 0$, so $\lambda = 9$.

40. (D) We must have $a = 2^r 3^s 5^t$ and $b = 2^u 3^v 5^w$, where $0 \le r \le u \le 3$, $0 \le s \le v \le 3$. and $0 \le t \le w \le 3$. There are 10 ways to choose (r, u) ($\binom{4}{2} = 6$ ways to choose two distinct integers, and 4 ways to choose the same integer twice), and similarly for (s, v) and (t, w). These gives a total of $10^3 = 1000$ pairs (a, b), but we are counting the $4^3 = 64$ cases where $a = b$. $1000 - 64 = 936$.