MMPC STUDENT CODE

THE SIXTY-FIRST ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America

Part II

Wednesday, December 6, 2017

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. **PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.**

2. Part II consists of problems and proofs. You will be allowed 100 minutes (1 hour and 40 minutes) for the five questions. To receive full credit for a problem, you are expected to justify your answer.

3. You are not expected to solve all problems completely. Look over all the problems and work first on those that interest you the most. If you are unable to solve a particular problem, partial credit might be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.

4. Each problem is on a separate page. If possible, you should show all of your work on that page. If there is not enough room, you may continue your solution on the blank page at the end of the booklet (page 7) or on additional paper inserted into the examination booklet. Be certain to **check the appropriate box** to report where your continuation occurs. On the continuation page clearly write the **problem number**. If you use additional paper for your answer, check the appropriate box and write your **identification number** and the **problem number** in the upper right-hand corner of each additional sheet.

5. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate an idea of how to solve the problem. On the other hand, a carefully stated generalization may actually be easier to prove than the problem as stated.

6. The competition rules prohibit you from asking questions of anyone during the examination. The use of notes, reference material, computation aids, or any other aid is likewise prohibited. Please note that **calculators are not allowed** on this exam. When the supervisor announces that the 100 minutes are over, please cease work immediately and insert all significant extra paper into the test booklet. Please do not return scratch paper containing routine numerical calculations.

7. You may now open the test booklet.

#	1	2	3	4	5	Total
Score						

1. Consider a normal 8×8 chessboard, where each square is labelled with either 1 or -1. Let a_k be the product of the numbers in the kth row, and let b_k be the product of the numbers in the kth column. Find, with proof, all possible values of $\sum_{k=1}^{8}(a_k b_k)$.

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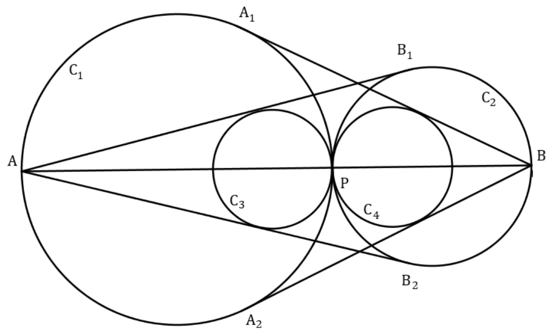
2. Let $\overline{AB}$ be a line segment with $AB = 1$, and P be a point on $\overline{AB}$ with $AP = x$, for some $0 < x < 1$. Draw circles C_1 and C_2 with $\overline{AP}$, $\overline{PB}$ as diameters, respectively. Let $\overline{AB_1}$, $\overline{AB_2}$ be tangent to C_2 at B_1 and B_2, and let $\overline{BA_1}, \overline{BA_2}$ be tangent to C_1 at A_1 and A_2. Now C_3 is a circle tangent to C_2, $\overline{AB_1}$, and $\overline{AB_2}$; C_4 is a circle tangent to C_1, $\overline{BA_1}$, and $\overline{BA_2}$.



(a) Express the radius of C_3 as a function of x.

(b) Prove that C_3 and C_4 are congruent.

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3. Suppose that the graphs of $y = (x + a)^2$ and $x = (y + a)^2$ are tangent to one another at a point on the line $y = x$. Find all possible values of a.

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4. You may assume without proof or justification that the infinite radical expressions

$$\sqrt{a-\sqrt{a-\sqrt{a-\sqrt{a-\cdots}}}}$$

and

$$\sqrt{a-\sqrt{a+\sqrt{a-\sqrt{a+\cdots}}}}$$

represent unique values for $a>2$.

(a) Find a real number a such that

$$\sqrt{a-\sqrt{a-\sqrt{a-\sqrt{a-\cdots}}}}=2017.$$

(b) Show that

$$\sqrt{2018-\sqrt{2018+\sqrt{2018-\sqrt{2018+\cdots}}}}=\sqrt{2017-\sqrt{2017-\sqrt{2017-\sqrt{2017-\cdots}}}}.$$

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5. (a) Suppose that m, n are positive integers such that $7n^2 - m^2 > 0$. Prove that, in fact, $7n^2 - m^2 \geq 3$.
   (b) Suppose that m, n are positive integers such that $\frac{m}{n} < \sqrt{7}$. Prove that, in fact, $\frac{m}{n} + \frac{1}{mn} < \sqrt{7}$.

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(Continued Solutions)

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America

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ACKNOWLEDGMENTS

- We wish to thank Mu Alpha Theta for its financial assistance.
- We wish to thank Northern Michigan University for its support in hosting this competition.
- We wish to thank Albion College for hosting the MMPC Grading Day and the Banquet.