THE SIXTY-FIRST ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

Sponsored by

The Michigan Section of the Mathematical Association of America

Part I

Tuesday, October 10, 2017

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Carefully read and follow the instructions printed on the answer sheet. Check to ensure that your five-digit MMPC code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor asks you to stop, please quit working immediately and turn in your answer sheet.

3. Consider the problems and responses carefully. You may work out ideas on scratch paper before selecting a response.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number of correct answers. You are advised to guess an answer in those cases where you cannot determine an answer.

5. For each of the questions, five different possible responses are provided. Choose the correct answer and completely fill in the corresponding bubble on your answer sheet.

6. Any scientific or graphing calculator is permitted on Part I. Unacceptable machines include computers, PDAs, pocket organizers, cell phones, and similar devices. All problems will be solvable with no more technology than a scientific calculator. The Exam Committee makes every effort to structure the test to minimize the advantage of a more powerful calculator. No other devices are permitted.

7. No one is permitted to explain to you the meaning of any question. Do not ask anyone to violate the rules of the competition. If you have questions concerning the instructions, ask them now.

8. You may open the test booklet and begin.

1. Oscar scores 80 on his first exam in a class. After each exam he takes, his average exam score increases by 1 point. What score did Oscar receive on his fifth exam?

(A) 84 (B) 85 (C) 86 (D) 88 (E) 90

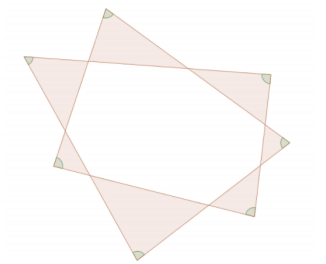
2. The Hypothetical Animal Shelter is home to many cats and dogs. Currently 60% of the animals are cats, and the other 40% are dogs. If there were 10% fewer cats and 35% more dogs, what percentage of the animals would be cats?

(A) 45% (B) 50% (C) 54% (D) 56% (E) 64%

3. The sum of 2017 consecutive integers is 2017. What is the largest number?

(A) 1008 (B) 1009 (C) 1135 (D) 1203 (E) 2017

4. What is the sum of the measures of the 7 marked angles of the seven-pointed star below?



(A) 180° (B) 360° (C) 540° (D) 720° (E) 900°

5. If you write the following number (in base ten), how many digits does it have?

$$|(2017)(201720172017) - (20172017)(20172017)|$$

(A) 11 (B) 12 (C) 13 (D) 14 (E) 15

6. We call a triangle *fantastic* if $2\alpha + \beta = \gamma$, where α, β, γ are the angles of the triangle. Which of the following is true about fantastic triangles?

(A) All are acute. (B) All are right. (C) All are obtuse.
(D) Some are right, some are acute. (E) Some are right, some are obtuse.

7. How many fractions $\frac{n}{700}$, where n is an integer such that $0 < n < 699$, can also be written as a terminating decimal?

(A) 7 (B) 24 (C) 49 (D) 70 (E) 99

8. There is a pile of pennies, nickels, dimes, and quarters. (These are coins worth 1 cent, 5 cents, 10 cents, and 25 cents, respectively.) If the total value of all the pennies, the total value of all the nickels, the total value of all the dimes, and the total value of all the quarters are all equal, which of the following could be the total number of coins?

(A) 335 (B) 342 (C) 349 (D) 356 (E) 363

9. The diagonals of convex quadrilateral $ABCD$ are perpendicular. If $AB = 9, BC = 5, CD = 12$, then what is the length of $\overline{AD}$?

(A) 11 (B) $7\sqrt{3}$ (C) 14 (D) $10\sqrt{2}$ (E) 16

10. The function f is defined on the domain $x < 0$ by $f(x) = (x-1)^2$. What is the inverse of f?

(A) $f^{-1}(x) = 1 + \sqrt{x}, x > 1$ (B) $f^{-1}(x) = 1 - \sqrt{x}, x > 1$
(C) $f^{-1}(x) = \sqrt{x-1}, x < 1$ (D) $f^{-1}(x) = \sqrt{x+1}, x > 0$
(E) $f^{-1}(x) = 1 + \sqrt{|x|}, x < 0$

11. How many of the positive integers $1, 2, 3, \ldots, 2017$ can be expressed in the form $\frac{\sqrt[3]{n}}{\sqrt[5]{n}}$ for some positive integer n?

(A) 4 (B) 12 (C) 32 (D) 44 (E) 672

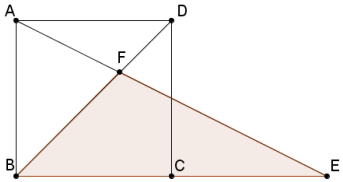
12. Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -x + 2017$. What value of x satisfies the following equation?

$$\underbrace{(f \circ f \circ \cdots \circ f)}_{2017 \text{ times}}(x) = x$$

(Note. Here $f \circ f$ denotes the composition of functions: $(f \circ f)(x) = f(f(x))$. Similarly $(f \circ f \circ f)(x) = f(f(f(x)))$, and so on.)

(A) 0 (B) $\sqrt{2017}$ (C) 2017/2
(D) 2017 (E) There is more than one value.

13. Let $ABCD$ be a square with side length 1, E be a point on the extension of $\overline{BC}$ with $BC = CE$, and F be the intersection point of $\overline{AE}$ and $\overline{BD}$, as in the graph below.



What is the area of $\triangle BEF$?

(A) $\frac{1}{2}$ (B) $\frac{3}{5}$ (C) $\frac{2}{3}$ (D) $\frac{3}{4}$ (E) $\frac{4}{5}$

14. The points $(1, 2, 4)$, $(2, 3, 6)$, and $(3, 5, 9)$ lie on a plane in three-dimensional space. Which of these other points also lies on this plane?

(A) $(-1, -1, -1)$ (B) $(0, 0, 2)$ (C) $(4, -2, 1)$
(D) $(5, 8, 13)$ (E) $(20, 17, 2017)$

15. For what real values of x is $\log_2(9 - 2^x) > 3 - x$?

(A) $0 < x < 3$ (B) $x < 1$ (C) $1 < x < 8$
(D) $0 < x < \log_2 9$ (E) $1 < x < \log_2 9$

16. How many ordered quadruples of distinct integers (a, b, c, d) satisfy the condition $abcd = 10$?

(A) 12 (B) 24 (C) 48 (D) 96 (E) 144

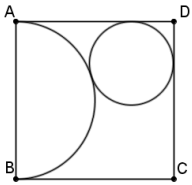
17. In a rectangular prism, the surface area is 28 square units, and the long diagonal (passing through the interior of the solid) has length 6 units. What is the sum of the lengths of the 12 edges of the solid?

(A) 24 (B) 32 (C) 40
(D) More than one answer is possible. (E) Such a prism cannot exist.

18. What point is the reflection of $(6, 2)$ over the line $y = 2x$?

(A) $(0, 4)$ (B) $(-3, 5)$ (C) $(2, 6)$ (D) $(-2, 6)$ (E) $(-1, 4)$

19. Let $ABCD$ be a square with side length 1, draw a semicircle inside the square with $\overline{AB}$ as its diameter, and draw a circle inside the square that is tangent to the semicircle, and tangent to the sides $\overline{AD}$ and $\overline{CD}$, as in the graph below:



What is the radius of the circle?

(A) 1/4 (B) $2-\sqrt{3}$ (C) $\frac{2-\sqrt{2}}{2}$ (D) $\sqrt{3}-\sqrt{2}$ (E) $\frac{\sqrt{5}+1}{4}$

20. A parallelogram of area equal to 12 square units has two vertices at the points $(2,1)$ and $(5,-2)$, respectively. If the intersection of the diagonals of the parallelogram is on the positive x-axis, which of these could be another vertex of the parallelogram?

(A) $(-2,0)$ (B) $(0,-2)$ (C) $(1,0)$ (D) $(5,2)$ (E) $(6,3)$

21. Form a sequence $b_1 = b_2 = 1$, and $b_n = b_{n-1}^2 + b_{n-2}$ for $n > 2$. How many of the first 2017 terms are multiples of 3?

(A) 504 (B) 672 (C) 673 (D) 756 (E) 757

22. A line parallel to $y = 3x$ intersects the parabola $y = x^2 - 3x + 6$ at two points A, B. Let M be the midpoint of $\overline{AB}$. What is the x-coordinate of M?

(A) 0 (B) 1 (C) 1.5 (D) 2 (E) 3

23. In triangle ABC, $AB = AC = 5$ and $BC = 6$. Let O be the center of the circle through A, B, C. What is the distance between O and $\overline{AB}$?

(A) 8/5 (B) 5/3 (C) $\sqrt{15}/2$ (D) 15/8 (E) 2

24. Which best describes the graph of $||x+y| - |x-y|| = 1$?

(A) four rays (B) eight rays (C) four lines
(D) one circle (E) two circles

25. A regular hexagon and an equilateral triangle have the same area. What is the ratio of the perimeter of the hexagon to the perimeter of the triangle?

(A) $2/\sqrt{6}$ (B) 3/2 (C) $\sqrt{3}$ (D) $\sqrt{6}/2$ (E) $6/\sqrt{3}$

26. If we expand $(\sqrt[3]{2} - \sqrt[5]{3})^{14}$ using the Binomial Theorem, exactly one of the 15 terms is an integer. What is this term?

(A) $\binom{14}{10}$ (B) $2\binom{14}{3}$ (C) $-24\binom{14}{5}$ (D) $-6\binom{14}{8}$ (E) $-12\binom{14}{11}$

27. If a, b, c are positive integers such that $\gcd(a, b) = 700$ and $\gcd(a, c) = 60$, which of these is a possible value for $\gcd(b, c)$?

(A) 100 (B) 130 (C) 180 (D) 220 (E) 560

28. How many ways are there to express $1/2$ in the form

$$\frac{1}{2} = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k},$$

where $1 < a_1 < a_2 < \ldots < a_k$ are integers?

(A) 2 (B) 4 (C) 6 (D) 8 (E) infinitely many

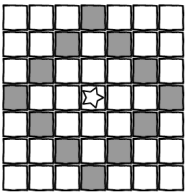
29. If i is a square root of -1, what is $1 + i + i^2 + i^3 + \cdots + i^{2017}$?

(A) i (B) 2 (C) $1 + i$ (D) $1 - i$ (E) $1/(1 + i)$

30. If α, β are acute angles such that $\tan\alpha = \frac{1+\tan\beta}{1-\tan\beta}$, then what must be true about α, β?

(A) $\alpha = \beta$ (B) $\alpha + \beta = 45^\circ$ (C) $\alpha - \beta = 45^\circ$
(D) $\alpha \pm \beta = 60^\circ$ (E) $\alpha + \beta = 90^\circ$

31. A token begins at the starred square of the grid shown. At each move, the piece moves up, down, left, or right to an adjacent square with equal probability. What is the probability that the token is on a shaded square after three moves?



(A) 1/4 (B) 5/16 (C) 1/3 (D) 25/64 (E) 7/16

32. Define a sequence by $c_1 = 1$ and, for $n > 1$, $c_n = \frac{n^2}{n-1} c_{n-1}$. What is the sum $c_1 + c_2 + \cdots + c_{2017}$?

(A) $2017! - 1$ (B) $2017!$ (C) $2018!/2$ (D) $2018! - 1$ (E) $2018!$

33. What is the area of a hexagon with all angles congruent and side lengths alternately 1 and 7?

(A) $\frac{39\sqrt{3}}{2}$ (B) $20\sqrt{3}$ (C) $\frac{61\sqrt{3}}{3}$ (D) $\frac{62\sqrt{3}}{3}$ (E) $\frac{43\sqrt{3}}{2}$

34. When the polynomial $p(x)$ is divided by $x^2 - 1$, the remainder is $x + 3$. Which of these might be the remainder when $p(x)$ is divided by $x^3 - 1$?

(A) 6 (B) $x + 2$ (C) $3x + 1$ (D) $x^2 + x - 1$ (E) $x^2 + 4x + 3$

35. How many ordered 5-tuples $(a_1, a_2, a_3, a_4, a_5)$ have the property that $a_i \in \{-1, 0, 1\}$ and for $1 \leq i \leq 4$, $a_i a_{i+1} \geq 0$?

(A) 35 (B) 48 (C) 64 (D) 99 (E) 128

36. Anjali and Bao each have a fair six-sided die. The faces of Anjali's die are labelled with the numbers from 1 to 6 as normal; the faces of Bao's die are labelled with the numbers from 2 to 7. Each rolls her die, and the higher number wins. If both dice show the same number, they roll again, repeating as necessary. What is the probability that Anjali will win?

(A) 7/31 (B) 1/4 (C) 11/36 (D) 1/3 (E) 10/31

37. For what values of the real parameter m is the following function f one-to-one on the domain of all real numbers?

$$f(x) = \begin{cases} -x^2 + 2mx + 1 & \text{if } x \leq 0 \\ mx + 1 & \text{if } x > 0 \end{cases}$$

(A) $m < 0$ (B) $m > 0$ (C) $m = 0$ (D) $m \neq 0$ (E) any real m

38. How many real solutions does the following equation have?

$$24^x - 3 \cdot 12^x - 2 \cdot 8^x - 9 \cdot 6^x + 18 \cdot 2^x + 27 \cdot 3^x + 6 \cdot 4^x - 54 = 0$$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

39. Two of the three roots of the equation $2x^3 - 4x^2 - 7x + \lambda = 0$ add up to 1. What is λ?

(A) -1 (B) 0 (C) 1 (D) 6 (E) 9

40. How many ordered pairs of distinct positive integers (a, b) have the property that a divides b and b divides 27000?

(A) 216 (B) 624 (C) 720 (D) 936 (E) 1000

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

MMPC DIRECTOR

Andy Poe
Northern Michigan University

MICHIGAN SECTION EXECUTIVE COMMITTEE

Governor
Steve Schlicker, GVSU

Chair
Laura McLeman, UM, Flint

Vice Chairs
Victor Piercey, Ferris State
Natascha Rivet, Delta College

Secretary-Treasurer
Mark Bollman, Albion College

Past Chair
Gavin LaRose, UM, Ann Arbor

EXAMINATION COMMITTEE

Chair
Michael Cap Khoury
Everi Games

David Friday
Macomb CC

Dorin Dumitrascu
Adrian College

Bingwu Wang
Eastern Michigan University

ACKNOWLEDGMENTS

- The Michigan Association of Secondary School Principals has placed this competition on the Approved List of Michigan Contests and Activities.
- We wish to thank Northern Michigan University for its support in hosting the competition.
- We wish to thank Albion College for hosting the MMPC grading day.